31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disc:
$$\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2)\, dy = 16\pi$$
$$\frac{1}{9}\int_{-3}^{y_0} (9 - y^2)\, dy = 1$$
$$\left[9y - \frac{1}{3}y^3\right]_{-3}^{y_0} = 9$$
$$\left(9y_0 - \frac{1}{3}y_0^3\right) - (-27 + 9) = 9$$
$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

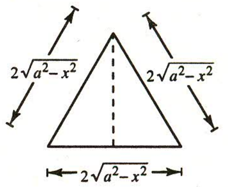
32. $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{a^2 - x^2}\right)\left(\sqrt{3}\sqrt{a^2 - x^2}\right)$

$= \sqrt{3}(a^2 - x^2)$

$$V = \sqrt{3}\int_{-a}^{a} (a^2 - x^2)\, dx = \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$
$$= \sqrt{3}\left(\frac{4a^3}{3}\right)$$

Since $\left(4\sqrt{3}\,a^3\right)/3 = 10$, we have $a^3 = \left(5\sqrt{3}\right)/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



33.
$$f(x) = \frac{4}{5}x^{5/4}$$
$$f'(x) = x^{1/4}$$
$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$
$$u = 1 + \sqrt{x}$$
$$x = (u - 1)^2$$
$$dx = 2(u - 1)\, du$$
$$s = \int_0^4 \sqrt{1 + \sqrt{x}}\, dx = 2\int_1^3 \sqrt{u}(u - 1)\, du$$
$$= 2\int_1^3 (u^{3/2} - u^{1/2})\, du$$
$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^3 = \frac{4}{15}\left[u^{3/2}(3u - 5)\right]_1^3$$
$$= \frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$$

34.
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$
$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2$$
$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^3 = \frac{14}{3}$$

35. $y = 300\cosh\left(\frac{x}{2000}\right) - 280,\ -2000 \le x \le 2000$

$$y' = \frac{3}{20}\sinh\left(\frac{x}{2000}\right)$$
$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20}\sinh\left(\frac{x}{2000}\right)\right]^2}\, dx = \frac{1}{20}\int_{-2000}^{2000} \sqrt{400 + 9\sinh^2\left(\frac{x}{2000}\right)}\, dx$$

≈ 4018.2 ft (by Simpson's Rule or graphing utility)

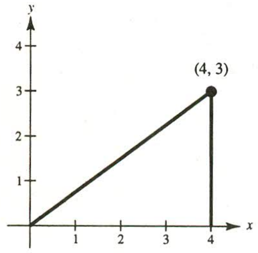
36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right)\sqrt{\frac{25}{16}}\,dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx = 4\pi \int_0^3 \sqrt{x+1}\,dx$$

$$= 4\pi\left[\left(\frac{2}{3}\right)(x+1)^{3/2}\right]_0^3 = \frac{56\pi}{3}$$

39. $F = kx$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x\,dx = \Big[2x^2\Big]_0^5$$

$$= 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x\,dx = \left[\frac{25}{9}x^2\right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

41. Volume of disc: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disc: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9}\int_0^{150} (175 - y)\,dy = \frac{62.4\pi}{9}\left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104{,}000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3\text{/min}$$

$$V = \pi r^2 h = \pi\left(\frac{1}{9}\right)h$$

$$\frac{dV}{dt} = \frac{\pi}{9}\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{9}{\pi}\left(\frac{dV}{dt}\right) = \frac{9}{\pi}\left(-\frac{8}{7.481}\right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49$ minutes

$(49)(12) = 588$ gallons pumped

Volume of water pumped in Exercise 41: 391.7 gallons

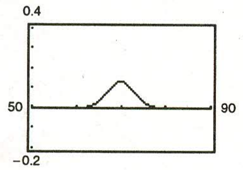
$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi$ ft · ton

70. (a) $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$\int_{50}^{90} f(x)\,dx \approx 1.0$



(b) $P(72 \le x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \le x \le 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$P(70 \le x < \infty) = 0.5$

and

$$0.5 = P(70 \le x < \infty)$$
$$= P(70 \le x \le 72) + P(72 \le x < \infty).$$

71. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x\to\infty} 1/(x+1) = 0$, but

$$\int_0^{\infty} \frac{1}{x+1}\,dx = \lim_{b\to\infty} \Big[\ln|x+1|\Big]_0^b = \infty.$$

Diverges

72. False. This is equivalent to Exercise 71.

73. True

74. True

Review Exercises for Chapter 7

1.
$$\int e^{2x} \sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$
$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$$
$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$
$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \Rightarrow v = -\frac{1}{3}\cos 3x$

$u = e^{2x} \Rightarrow du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \Rightarrow v = \frac{1}{3}\sin 3x$

$u = e^{2x} \Rightarrow du = 2e^{2x}\,dx$

2. $\int (x^2-1)e^x\,dx = (x^2-1)e^x - 2\int xe^x\,dx = (x^2-1)e^x - 2xe^x + 2\int e^x\,dx = e^x(x^2-2x+1) + 1$

(1) $dv = e^x\,dx \Rightarrow v = e^x$

$u = x^2 - 1 \Rightarrow du = 2x\,dx$

(2) $dv = e^x\,dx \Rightarrow v = e^x$

$u = x \Rightarrow du = dx$

13. $\displaystyle\int \frac{1}{1-\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$

14. $\displaystyle\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$

$\displaystyle = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$

$\displaystyle = -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

(1) $\displaystyle dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x^2 \implies du = 2x\,dx$

(2) $\displaystyle dv = \cos 2x\,dx \implies v = \frac{1}{2}\sin 2x$

$u = x \implies du = dx$

15. $\displaystyle\int \frac{\ln 2x}{x^2}\,dx = \frac{-\ln 2x}{x} + \int \frac{1}{x^2}\,dx$

$\displaystyle = \frac{-\ln 2x}{x} - \frac{1}{x} + C$

$\displaystyle = -\frac{1}{x}(1 + \ln 2x) + C$

$\displaystyle dv = \frac{1}{x^2}\,dx \implies v = -\frac{1}{x}$

$\displaystyle u = \ln 2x \implies du = \frac{1}{x}\,dx$

16. $\displaystyle\int 2x\sqrt{2x-3}\,dx = \int (u^4 + 3u^2)\,du = \frac{u^5}{5} + u^3 + C = \frac{2(2x-3)^{3/2}}{5}(x+1) + C$

$\displaystyle u = \sqrt{2x-3},\ x = \frac{u^2+3}{2},\ dx = u\,du$

17. $\displaystyle\int \sqrt{4-x^2}\,dx = \int (2\cos\theta)(2\cos\theta)\,d\theta$

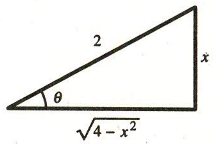
$\displaystyle = 2\int (1 + \cos 2\theta)\,d\theta$

$\displaystyle = 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$= 2(\theta + \sin\theta\cos\theta) + C$

$\displaystyle = 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$

$\displaystyle = \frac{1}{2}\left[4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right] + C$



$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta$

18. $\displaystyle\int \frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta = \int \frac{1}{u(u-1)}\,du = \int \frac{1}{u-1}\,du - \int \frac{1}{u}\,du$

$\displaystyle = \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$

$u = \tan\theta,\ du = \sec^2\theta\,d\theta$

$\displaystyle \frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + Bu$

Let $u = 0$: $1 = -A \implies A = -1$

Let $u = 1$: $1 = B$

19. $\dfrac{3x^3+4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$3x^3 + 4x = (Ax+B)(x^2+1) + Cx + D$

$= Ax^3 + Bx^2 + (A+C)x + (B+D)$

$A = 3, B = 0, A + C = 4 \Rightarrow C = 1,$

$B + D = 0 \Rightarrow D = 0$

$$\int \frac{3x^3+4x}{(x^2+1)^2}\,dx = 3\int \frac{x}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \frac{3}{2}\ln(x^2+1) - \frac{1}{2(x^2+1)} + C$$

20. $\displaystyle\int \sqrt{\frac{x-2}{x+2}}\,dx = \int \frac{x-2}{\sqrt{x^2-4}}\,dx$

$$= \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{1}{\sqrt{x^2-4}}\,dx$$

$$= \sqrt{x^2-4} - 2\ln\left|x + \sqrt{x^2-4}\right| + C$$

21. $\displaystyle\int \frac{16}{\sqrt{16-x^2}}\,dx = 16\arcsin\left(\frac{x}{4}\right) + C$

22. $\displaystyle\int \frac{\sin\theta}{1+2\cos^2\theta}\,d\theta = \frac{-1}{\sqrt{2}}\int \frac{1}{1+2\cos^2\theta}\left(-\sqrt{2}\sin\theta\right)d\theta$

$$= \frac{-1}{\sqrt{2}}\arctan\left(\sqrt{2}\cos\theta\right) + C$$

$u = \sqrt{2}\cos\theta,\ du = -\sqrt{2}\sin\theta\,d\theta$

23. $\displaystyle\int \frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\int \frac{2x+4-4}{x^2+4x+8}\,dx$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+8}\,dx - 2\int \frac{1}{(x+2)^2+4}\,dx = \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(\frac{x+2}{2}\right) + C$$

24. $\displaystyle\int \frac{x}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{2x-4+4}{x^2-4x+8}\,dx$

$$= \frac{1}{2}\int \frac{2x-4}{x^2-4x+8}\,dx + 2\int \frac{1}{(x-2)^2+4}\,dx = \frac{1}{2}\ln|x^2-4x+8| + \arctan\left(\frac{x-2}{2}\right) + C$$

25. $\displaystyle\int \theta\sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \theta\sin 2\theta\,d\theta$

$$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$$

$dv = \sin 2\theta\,d\theta \Rightarrow v = -\frac{1}{2}\cos 2\theta$

$u = \theta \Rightarrow du = d\theta$

26. $\displaystyle\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx = -\sqrt{2}\ln\left|\csc\sqrt{2x} + \cot\sqrt{2x}\right| + C$

$u = \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx$

27. $\displaystyle\int (\sin\theta + \cos\theta)^2\,d\theta = \int (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\,d\theta$

$$= \int (1 + \sin 2\theta)\,d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$$

65. (a) $\int_0^1 e^x\,dx = \Big[e^x\Big]_0^1 = e - 1 \approx 1.72$

(b) $\int_0^1 xe^x\,dx = \Big[e^x(x-1)\Big]_0^1 = 1.00$

(c) $\int_0^1 xe^{x^2}\,dx = \Big[\frac{1}{2}e^{x^2}\Big]_0^1 = \frac{1}{2}(e-1) \approx 0.86$

(d) Simpson's Rule ($n = 8$)

$$\int_0^1 e^{x^2}\,dx = \frac{1}{24}[1 + 4e^{(1/8)^2} + 2e^{(1/4)^2} + 4e^{(3/8)^2} + 2e^{(1/2)^2} + 4e^{(5/8)^2} + 2e^{(3/4)^2} + 4e^{(7/8)^2} + e] \approx 1.46$$

66. (a) $\int_0^{\pi/2} \cos x\,dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

(b) $\int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1 + \cos 2x)\,dx = \frac{1}{2}\Big[x + \sin x\cos x\Big]_0^{\pi/2} = \frac{\pi}{4} \approx 0.78$

(c) Simpson's Rule ($n = 4$)

$$\int_0^{\pi/2} \cos(x^2)\,dx = \frac{\pi}{24}\left[1 + 4\cos\left(\frac{\pi}{8}\right)^2 + 2\cos\left(\frac{\pi}{4}\right)^2 + 4\cos\left(\frac{3\pi}{8}\right)^2 + \cos\left(\frac{\pi}{2}\right)^2\right] \approx 0.85$$

(d) Simpson's Rule ($n = 4$)

$$\int_0^{\pi/2} \cos\sqrt{x}\,dx = \frac{\pi}{24}[1 + 4\cos\sqrt{\pi/8} + 2\cos\sqrt{\pi/4} + 4\cos\sqrt{3\pi/8} + \cos\sqrt{\pi/2}] \approx 1.01$$

67. $s = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx \approx 3.82$

68. $s = \int_0^{\pi} \sqrt{1 + \sin^2 2x}\,dx \approx 3.82$

69. $\lim_{x\to1}\left[\frac{(\ln x)^2}{x-1}\right] = \lim_{x\to1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$

70. $\lim_{x\to0}\frac{\sin \pi x}{\sin 2\pi x} = \lim_{x\to0}\frac{\pi\cos\pi x}{2\pi\cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$

71. $\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{4e^{2x}}{2} = \infty$

72. $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty}\frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$

73. $y = \lim_{x\to\infty}(\ln x)^{2/x}$

$\ln y = \lim_{x\to\infty}\frac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$

Since $\ln y = 0$, $y = 1$.

74. $y = \lim_{x\to1^+}(x-1)^{\ln x}$

$\ln y = \lim_{x\to1^+}[(\ln x)\ln(x-1)]$

$$= \lim_{x\to1^+}\left[\frac{\ln(x-1)}{\dfrac{1}{\ln x}}\right] = \lim_{x\to1^+}\left[\frac{\dfrac{1}{x-1}}{\left(\dfrac{1}{x}\right)\dfrac{-1}{\ln^2 x}}\right] = \lim_{x\to1^+}\left[\frac{-\ln^2 x}{\dfrac{x-1}{x}}\right] = \lim_{x\to1^+}\left[\frac{-2\left(\dfrac{1}{x}\right)(\ln x)}{\dfrac{1}{x^2}}\right]$$

$$= \lim_{x\to1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.