

31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disc: $\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2) dy = 16\pi$

$$\frac{1}{9} \int_{-3}^{y_0} (9 - y^2) dy = 1$$

$$\left[9y - \frac{1}{3}y^3 \right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \frac{1}{3}y_0^3 \right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

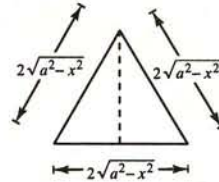
By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958$ ft.

$$\begin{aligned} 32. A(x) &= \frac{1}{2}bh = \frac{1}{2}(2\sqrt{a^2 - x^2})(\sqrt{3}\sqrt{a^2 - x^2}) \\ &= \sqrt{3}(a^2 - x^2) \end{aligned}$$

$$\begin{aligned} V &= \sqrt{3} \int_{-a}^a (a^2 - x^2) dx = \sqrt{3} \left[a^2x - \frac{x^3}{3} \right]_{-a}^a \\ &= \sqrt{3} \left(\frac{4a^3}{3} \right) \end{aligned}$$

Since $(4\sqrt{3}a^3)/3 = 10$, we have $a^3 = (5\sqrt{3})/2$. Thus,

$$a = \sqrt[3]{\frac{5\sqrt{3}}{2}} \approx 1.630 \text{ meters.}$$



33. $f(x) = \frac{4}{5}x^{5/4}$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u - 1)^2$$

$$dx = 2(u - 1) du$$

$$s = \int_0^4 \sqrt{1 + \sqrt{x}} dx = 2 \int_1^3 \sqrt{u}(u - 1) du$$

$$= 2 \int_1^3 (u^{3/2} - u^{1/2}) du$$

$$= 2 \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^3 = \frac{4}{15} \left[u^{3/2}(3u - 5) \right]_1^3$$

$$= \frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076$$

34. $y = \frac{x^3}{6} + \frac{1}{2x}$

$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right)^2$$

$$s = \int_1^3 \left(\frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{2x} \right]_1^3 = \frac{14}{3}$$

35. $y = 300 \cosh\left(\frac{x}{2000}\right) - 280, -2000 \leq x \leq 2000$

$$y' = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$$

$$s = \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20} \sinh\left(\frac{x}{2000}\right) \right]^2} dx = \frac{1}{20} \int_{-2000}^{2000} \sqrt{400 + 9 \sinh^2\left(\frac{x}{2000}\right)} dx$$

$$\approx 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

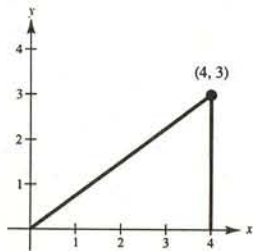
36. Since $f(x) = \tan x$ has $f'(x) = \sec^2 x$, this integral represents the length of the graph of $\tan x$ from $x = 0$ to $x = \pi/4$. This

37. $y = \frac{3}{4}x$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right) \sqrt{\frac{25}{16}} dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



39. $F = kx$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x dx = \left[2x^2\right]_0^5$$

$$= 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

41. Volume of disc: $\pi\left(\frac{1}{3}\right)^2 \Delta y$

Weight of disc: $62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$

Distance: $175 - y$

$$W = \frac{62.4\pi}{9} \int_0^{150} (175 - y) dy = \frac{62.4\pi}{9} \left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104,000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

38. $y = 2\sqrt{x}$

$$y' = \frac{1}{\sqrt{x}}$$

$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$S = 2\pi \int_0^3 2\sqrt{x} \sqrt{\frac{x+1}{x}} dx = 4\pi \int_0^3 \sqrt{x+1} dx$$

$$= 4\pi \left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = \frac{56\pi}{3}$$

40. $F = kx$

$$50 = k(9) \Rightarrow k = \frac{50}{9}$$

$$F = \frac{50}{9}x$$

$$W = \int_0^9 \frac{50}{9}x dx = \left[\frac{25}{9}x^2\right]_0^9$$

$$= 225 \text{ in} \cdot \text{lb} = 18.75 \text{ ft} \cdot \text{lb}$$

42. We know that

$$\frac{dV}{dt} = \frac{4 \text{ gal/min} - 12 \text{ gal/min}}{7.481 \text{ gal/ft}^3} = -\frac{8}{7.481} \text{ ft}^3/\text{min}$$

$$V = \pi r^2 h = \pi\left(\frac{1}{9}\right)h$$

$$\frac{dV}{dt} = \frac{\pi}{9} \left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} = \frac{9}{\pi} \left(\frac{dV}{dt}\right) = \frac{9}{\pi} \left(-\frac{8}{7.481}\right) \approx -3.064 \text{ ft/min.}$$

Depth of water: $-3.064t + 150$

Time to drain well: $t = \frac{150}{3.064} \approx 49$ minutes

$$(49)(12) = 588 \text{ gallons pumped}$$

Volume of water pumped in Exercise 41: 391.7 gallons

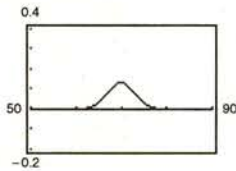
$$\frac{391.7}{52\pi} = \frac{588}{x\pi}$$

$$x = \frac{588(52)}{391.7} \approx 78$$

Work $\approx 78\pi \text{ ft} \cdot \text{ton}$

70. (a) $f(x) = \frac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x) dx \approx 1.0$$



(b) $P(72 \leq x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \leq x \leq 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

$$P(70 \leq x < \infty) = 0.5$$

and

$$0.5 = P(70 \leq x < \infty)$$

$$= P(70 \leq x \leq 72) + P(72 \leq x < \infty).$$

71. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but

$$\int_0^{\infty} \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty.$$

Diverges

72. False. This is equivalent to Exercise 71.

73. True

74. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$2. \int (x^2 - 1)e^x dx = (x^2 - 1)e^x - 2 \int xe^x dx = (x^2 - 1)e^x - 2xe^x + 2 \int e^x dx = e^x(x^2 - 2x + 1) + 1$$

$$(1) dv = e^x dx \Rightarrow v = e^x$$

$$u = x^2 - 1 \Rightarrow du = 2x dx$$

$$(2) dv = e^x dx \Rightarrow v = e^x$$

$$u = x \Rightarrow du = dx$$

$$13. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$14. \int x^2 \sin 2x dx = -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx$$

$$= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$15. \int \frac{\ln 2x}{x^2} dx = \frac{-\ln 2x}{x} + \int \frac{1}{x^2} dx$$

$$= \frac{-\ln 2x}{x} - \frac{1}{x} + C$$

$$= -\frac{1}{x}(1 + \ln 2x) + C$$

$$dv = \frac{1}{x^2} dx \Rightarrow v = -\frac{1}{x}$$

$$u = \ln 2x \Rightarrow du = \frac{1}{x} dx$$

$$16. \int 2x\sqrt{2x-3} dx = \int (u^4 + 3u^2) du = \frac{u^5}{5} + u^3 + C = \frac{2(2x-3)^{3/2}}{5}(x+1) + C$$

$$u = \sqrt{2x-3}, x = \frac{u^2+3}{2}, dx = u du$$

$$17. \int \sqrt{4-x^2} dx = \int (2 \cos \theta)(2 \cos \theta) d\theta$$

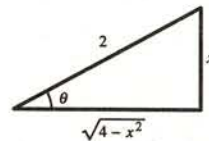
$$= 2 \int (1 + \cos 2\theta) d\theta$$

$$= 2\left(\theta + \frac{1}{2} \sin 2\theta\right) + C$$

$$= 2(\theta + \sin \theta \cos \theta) + C$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$$

$$= \frac{1}{2}\left[4 \arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right] + C$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

$$18. \int \frac{\sec^2 \theta}{\tan \theta (\tan \theta - 1)} d\theta = \int \frac{1}{u(u-1)} du = \int \frac{1}{u-1} du - \int \frac{1}{u} du$$

$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan \theta - 1}{\tan \theta}\right| + C = \ln|1 - \cot \theta| + C$$

$$u = \tan \theta, du = \sec^2 \theta d\theta$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

$$\text{Let } u = 0: 1 = -A \Rightarrow A = -1$$

$$\text{Let } u = 1: 1 = B$$

$$19. \frac{3x^3 + 4x}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

$$3x^3 + 4x = (Ax + B)(x^2 + 1) + Cx + D$$

$$= Ax^3 + Bx^2 + (A + C)x + (B + D)$$

$$A = 3, B = 0, A + C = 4 \Rightarrow C = 1,$$

$$B + D = 0 \Rightarrow D = 0$$

$$\int \frac{3x^3 + 4x}{(x^2 + 1)^2} dx = 3 \int \frac{x}{x^2 + 1} dx + \int \frac{x}{(x^2 + 1)^2} dx$$

$$= \frac{3}{2} \ln(x^2 + 1) - \frac{1}{2(x^2 + 1)} + C$$

$$21. \int \frac{16}{\sqrt{16 - x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$20. \int \sqrt{\frac{x-2}{x+2}} dx = \int \frac{x-2}{\sqrt{x^2-4}} dx$$

$$= \int \frac{x}{\sqrt{x^2-4}} dx - 2 \int \frac{1}{\sqrt{x^2-4}} dx$$

$$= \sqrt{x^2-4} - 2 \ln|x + \sqrt{x^2-4}| + C$$

$$22. \int \frac{\sin \theta}{1 + 2 \cos^2 \theta} d\theta = \frac{-1}{\sqrt{2}} \int \frac{1}{1 + 2 \cos^2 \theta} (-\sqrt{2} \sin \theta) d\theta$$

$$= \frac{-1}{\sqrt{2}} \arctan(\sqrt{2} \cos \theta) + C$$

$$u = \sqrt{2} \cos \theta, du = -\sqrt{2} \sin \theta d\theta$$

$$23. \int \frac{x}{x^2 + 4x + 8} dx = \frac{1}{2} \int \frac{2x + 4 - 4}{x^2 + 4x + 8} dx$$

$$= \frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx - 2 \int \frac{1}{(x + 2)^2 + 4} dx = \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan\left(\frac{x + 2}{2}\right) + C$$

$$24. \int \frac{x}{x^2 - 4x + 8} dx = \frac{1}{2} \int \frac{2x - 4 + 4}{x^2 - 4x + 8} dx$$

$$= \frac{1}{2} \int \frac{2x - 4}{x^2 - 4x + 8} dx + 2 \int \frac{1}{(x - 2)^2 + 4} dx = \frac{1}{2} \ln|x^2 - 4x + 8| + \arctan\left(\frac{x - 2}{2}\right) + C$$

$$25. \int \theta \sin \theta \cos \theta d\theta = \frac{1}{2} \int \theta \sin 2\theta d\theta$$

$$= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$26. \int \frac{\csc \sqrt{2x}}{\sqrt{x}} dx = \sqrt{2} \int \csc \sqrt{2x} \left(\frac{1}{\sqrt{2x}}\right) dx = -\sqrt{2} \ln|\csc \sqrt{2x} + \cot \sqrt{2x}| + C$$

$$u = \sqrt{2x}, du = \frac{1}{\sqrt{2x}} dx$$

$$27. \int (\sin \theta + \cos \theta)^2 d\theta = \int (\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta) d\theta$$

$$= \int (1 + \sin 2\theta) d\theta = \theta - \frac{1}{2} \cos 2\theta + C = \frac{1}{2} (2\theta - \cos 2\theta) + C$$

$$65. (a) \int_0^1 e^x dx = [e^x]_0^1 = e - 1 \approx 1.72 \qquad (b) \int_0^1 xe^x dx = [e^x(x-1)]_0^1 = 1.00$$

$$(c) \int_0^1 xe^{x^2} dx = \left[\frac{1}{2} e^{x^2} \right]_0^1 = \frac{1}{2}(e - 1) \approx 0.86$$

(d) Simpson's Rule ($n = 8$)

$$\int_0^1 e^{x^2} dx = \frac{1}{24} [1 + 4e^{(1/8)^2} + 2e^{(1/4)^2} + 4e^{(3/8)^2} + 2e^{(1/2)^2} + 4e^{(5/8)^2} + 2e^{(3/4)^2} + 4e^{(7/8)^2} + e] \approx 1.46$$

$$66. (a) \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = 1$$

$$(b) \int_0^{\pi/2} \cos^2 x dx = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2x) dx = \frac{1}{2} [x + \sin x \cos x]_0^{\pi/2} = \frac{\pi}{4} \approx 0.78$$

(c) Simpson's Rule ($n = 4$)

$$\int_0^{\pi/2} \cos(x^2) dx = \frac{\pi}{24} \left[1 + 4 \cos\left(\frac{\pi}{8}\right)^2 + 2 \cos\left(\frac{\pi}{4}\right)^2 + 4 \cos\left(\frac{3\pi}{8}\right)^2 + \cos\left(\frac{\pi}{2}\right)^2 \right] \approx 0.85$$

(d) Simpson's Rule ($n = 4$)

$$\int_0^{\pi/2} \cos \sqrt{x} dx = \frac{\pi}{24} [1 + 4 \cos \sqrt{\pi/8} + 2 \cos \sqrt{\pi/4} + 4 \cos \sqrt{3\pi/8} + \cos \sqrt{\pi/2}] \approx 1.01$$

$$67. s = \int_0^{\pi} \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$68. s = \int_0^{\pi} \sqrt{1 + \sin^2 2x} dx \approx 3.82$$

$$69. \lim_{x \rightarrow 1} \frac{(\ln x)^2}{x-1} = \lim_{x \rightarrow 1} \left[\frac{2(1/x)\ln x}{1} \right] = 0$$

$$70. \lim_{x \rightarrow 0} \frac{\sin \pi x}{\sin 2\pi x} = \lim_{x \rightarrow 0} \frac{\pi \cos \pi x}{2\pi \cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$71. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$72. \lim_{x \rightarrow \infty} xe^{-x^2} = \lim_{x \rightarrow \infty} \frac{x}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{2xe^{x^2}} = 0$$

$$73. y = \lim_{x \rightarrow \infty} (\ln x)^{2/x}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0$$

Since $\ln y = 0$, $y = 1$.

$$74. y = \lim_{x \rightarrow 1^+} (x-1)^{\ln x}$$

$$\ln y = \lim_{x \rightarrow 1^+} [(\ln x) \ln(x-1)]$$

$$= \lim_{x \rightarrow 1^+} \left[\frac{\ln(x-1)}{\frac{1}{\ln x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x}\right) - 1} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-\ln^2 x}{\frac{x-1}{x}} \right] = \lim_{x \rightarrow 1^+} \left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}} \right]$$

$$= \lim_{x \rightarrow 1^+} 2x(\ln x) = 0$$

Since $\ln y = 0$, $y = 1$.