The Definite Integral and Riemann Sums

Using $f(x)=x^2$, find a right sum over the interval [2, 5] for the indicated number of rectangles:

n = 2 *n* = 4 *n* = 6

Notice that as we increase the number of rectangles Δx decreases AND our estimate for area becomes MORE ACCURATE!

Symbolically, $\frac{3}{n}$ represents the width of each rectangle(Δx) since the entire interval length is 3 and the number of rectangles used is n. So… $\frac{3}{n}=\Delta x$

Review: $\sum_{i=1}^{4} 2i+4=\left[2(1)+4\right]+\left[2(2)+4\right]+\left[2(3)+4\right]+\left[2(4)+4\right]=36$

$\sum_{i=1}^{n} f\left(2+\Delta x i\right)\bullet\Delta x$ **represents the right sum** since "2" is the left endpoint of the interval *and* using i =1, 2, 3… adds a factor of Δx to find each x-value evaluated in the function which finds rectangle heights.

Note: $\sum_{i=0}^{n-1} f\left(2+\Delta x i\right)\bullet\Delta x$ **represents the left sum** since starting with i = zero uses the left endpoint of the interval to find the first rectangle height.

When estimating area under curve using rectangles, the more of each used the better the estimate. Hypothetically, we could use and infinite amount of rectangles($n \to \infty$) meaning $\Delta x \to 0$ and the SUMS would approach the **EXACT** area.

So… $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+\frac{3}{n}i\right) \bullet \frac{3}{n}$ represents an EXACT area!

This EXACT value is called finding a **Riemann Sum** and is represented by the **Definite Integral** $\int_a^b f(x)\,dx$. "a" is called the lower limit and "b" is called the upper limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \bullet \Delta x = \int_a^b f(x)\,dx$$

AP Multiple Choice Practice

Which of the following limits is equal to $\int_2^5 x^2 dx$?

A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{k}{n}\right)^2 \frac{1}{n}$ B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{3k}{n}\right)^2 \frac{1}{n}$ C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{k}{n}\right)^2 \frac{3}{n}$ D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{3k}{n}\right)^2 \frac{3}{n}$

So… $\int_a^b f(x)\,dx$ means find the area between $f(x)$ and the x-axis from $x = a$ to $x = b$.

EX: Use geometry and properties of integration to evaluate each integral

A) $\int_3^6 4\,dx$ B) $\int_0^6 -|x| + 6\,dx$ C) $\int_{-4}^4 \sqrt{16-x^2}\,dx$

NOTE: When the y-values of $f(x)$ are BELOW the x-axis there is a problem because the y-values($f(x_i)$) would be NEGATIVE. Although area is ALWAYS positive, the integral answers could be negative(since multiplying negative y-values for the height create negative solutions).

EX: $\int_0^1 x-1\,dx$

Summary: Although area is ***always*** positive, a **definite integral** could be positive or negative depending on whether there is more area above or below the x-axis.

Note: If **area**(and not the integral) is actually is what is asked for, absolute value must be used to evaluate assuring all y-values will be positive giving a result of accumulated area.

Properties of Integration

If f is integrable on $[a, b]$, then for any constant "C" $\int_a^b \mathrm{C}f(x)\,dx = \mathrm{C}\int_a^b f(x)\,dx$

If f is integrable on $[a, b]$, then $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$

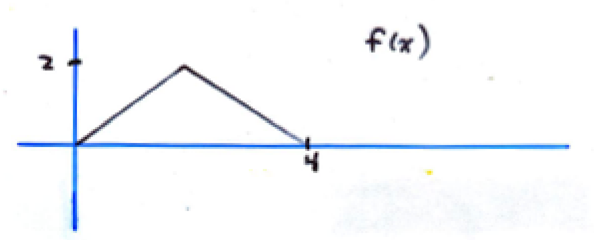
If f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If c is any value between $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

EX: $\int_0^2 x-1\,dx$ EX: $\int_0^6 |x-2|\,dx$ EX: $\int_6^0 |x-2|\,dx$

AP APPLICATIONS OF THE PROPERTIES OF INTEGRATION

Given the graph of $f(x)$



$\int_0^4 f(x)\,dx =$

Using Translations

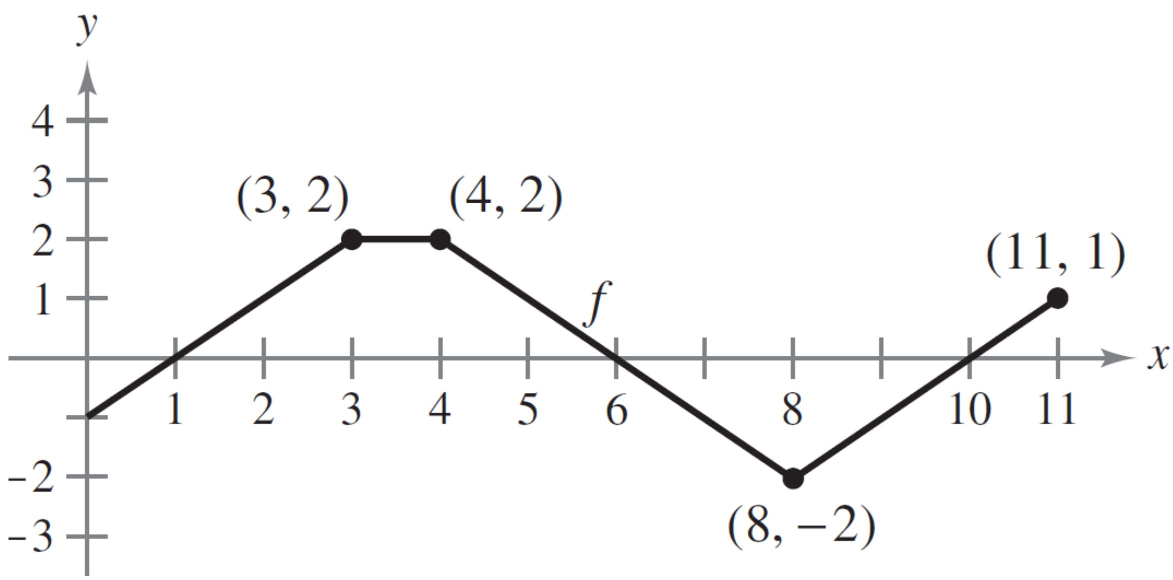
When the same interval length is used and the graph is translated left/right, area between the graph and the x-axis does not change so the integral is equivalent.

EX: $\int_{-2}^{2} f(x+2)\,dx$ EX: $\int_4^8 f(x-4)\,dx$

When the exact same interval is used and the graph is translated up/down(created by adding or subtracting a constant), ***use properties*** to separate the integral into two and evaluate each. Note: this is the only way to evaluate the integral since the area between the graph and the x-axis DOES change when graph is translated up/down.

EX: $\int_0^4 f(x)+3\,dx$ EX: $\int_0^4 f(x)-2\,dx$

EX



(a) $\int_0^1 -f(x)\,dx$

(b) $\int_3^4 3\,f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$

(d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$

(f) $\int_4^{10} f(x)\,dx$

(g) $\int_1^6 f(x)+3\,dx$

(h) $\int_4^8 f(x+2)\,dx$

CALCULATOR NOTE

Like derivatives, the definite integral can be found on the graphing calculator under 2nd, calc, $\int$ and use your lower and upper limit.

EX: Evaluate the following integral using the graphing calculator $\int_2^5 x^2\,dx$