Find the derivative of each function.

1a) $y = \ln\left(x\sqrt{x^2 - 1}\right)$

$y = \ln\left[x\sqrt{x^2 - 1}\right] = \ln x + \frac{1}{2}\ln\left(x^2 - 1\right)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x\left(x^2 - 1\right)}$

1b) $y = \ln(t + 1)^2$

$y = 2\ln(t+1)$

$y' = 2 \cdot \frac{1}{t+1} \cdot 1$

$y' = \frac{2}{t+1}$

1c) $f(x) = \ln\left(\frac{\sqrt{4 + x^2}}{x}\right)$

$f(x) = \ln(4+x^2)^{1/2} - \ln x$

$f(x) = \frac{1}{2}\ln(4+x^2) - \ln x$

$f'(x) = \frac{1}{2} \cdot \frac{1}{4+x^2} \cdot 2x - \frac{1}{x}$

$f'(x) = \frac{x}{4+x^2} - \frac{1}{x}$

1d) $g(t) = \frac{\ln t}{t^2}$

$g'(t) = \frac{t^2 \cdot \frac{1}{t} - \ln t \cdot (2t)}{(t^2)^2}$

$g'(t) = \frac{t - 2t \cdot \ln(t)}{t^4}$

$g'(t)\ \frac{t(1 - 2\ln t)}{t^{3}}$

$g'(t) = \frac{1 - 2\ln t}{t^3}$

1e) $y = \ln(\ln x^2)$

$y = \ln(2\ln x)$

$y' = \frac{1}{2\ln x} \cdot 2 \cdot \frac{1}{x}$

$y' = \frac{1}{x\ln x}$

Find the equation of the tangent line at the given point of each function.

2a) $f(x) = x^3 \ln x, \quad (1, 0)$

$f'(x) = 3x^2 \ln x + x^2$

$f'(1) = 1$

Tangent line: $y - 0 = 1(x - 1)$

$y = x - 1$

2b) $f(x) = \ln\sqrt{1 + \sin^2 x}, \quad \left(\frac{\pi}{4}, \ln\sqrt{\frac{3}{2}}\right)$

$f(x) = \frac{1}{2}\ln(1 + (\sin x)^2)$

$f'(x) = \frac{1}{2} \cdot \frac{1}{1+\sin^2 x} \cdot (0 + 2\sin x \cdot \cos x)$

$f'(x) = \frac{1}{2} \cdot \frac{\sin(2x)}{1+(\sin x)^2}$

$f'\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{\sin\left(\frac{\pi}{2}\right)}{1+\left(\sin\frac{\pi}{4}\right)^2}$

$f'\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{1}{1+\left(\frac{\sqrt{2}}{2}\right)^2}$

$f'\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{1}{\left(1+\frac{1}{2}\right)}$

$f'\left(\frac{\pi}{4}\right) = \frac{1}{2+1} = \frac{1}{3}$

So..

$y - \ln\sqrt{\frac{3}{2}} = \frac{1}{3}\left(x - \frac{\pi}{4}\right)$

Use implicit differentiation to find $\frac{dy}{dx}$.

3a) $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

3b) $4x^3 + \ln y^2 + 2y = 2x$ ($= 2\ln y$)

$$12x^2 + 2\cdot\frac{1}{y}\cdot\frac{dy}{dx} + 2\frac{dy}{dx} = 2$$

$$\frac{dy}{dx}\left(\frac{2}{y} + 2\right) = 2 - 12x^2$$

$$\frac{dy}{dx} = \frac{2 - 12x^2}{\frac{2}{y} + 2}\cdot\frac{y}{y}$$

$$= \frac{2y - 12x^2y}{2 + 2y} = \frac{2(y - 6x^2y)}{2(1+y)}$$

$$= \frac{y - 6x^2y}{1+y}$$

4a) **Find the equation of the tangent line at the given point of the function below.**

$x + y - 1 = \ln(x^2 + y^2)$, $(1, 0)$

$$1 + \frac{dy}{dx} - 0 = \frac{1}{x^2+y^2}\cdot\left(2x + 2y\frac{dy}{dx}\right)$$

$$1 + \frac{dy}{dx} = \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\frac{dy}{dx}$$

$$1 - \frac{2x}{x^2+y^2} = \frac{2y}{x^2+y^2}\frac{dy}{dx} - 1\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{2y}{x^2+y^2} - 1\right) = 1 - \frac{2x}{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{1 - \frac{2x}{x^2+y^2}}{\frac{2y}{x^2+y^2} - 1}$$

$$\frac{dy}{dx}(1,0) = \frac{1 - \frac{2}{1}}{\frac{0}{1} - 1} = \frac{-1}{-1} = 1$$

So....

$y - 0 = 1(x-1)$

or

$y = x - 1$

Use logarithms to help find $\frac{dy}{dx}$.

5a) $y = x\sqrt{x^2 + 1}$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2 + 1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 + 1}$$

$$\frac{dy}{dx} = y\left[\frac{2x^2 + 1}{x(x^2 + 1)}\right] = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

5b) $y = \frac{x^2\sqrt{3x - 2}}{(x + 1)^2}$

$$\ln y = \ln\frac{x^2\cdot(3x-2)^{1/2}}{(x+1)^2}$$

$$\ln y = \ln x^2 + \ln(3x-2)^{1/2} - \ln(x+1)^2$$

$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2}\cdot\frac{3}{3x-2} - \frac{2}{x+1}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{3}{6x-4} - \frac{2}{x+1}\right)$$

or

$$\frac{dy}{dx} = \frac{x^2(3x-2)^{1/2}}{(x+1)^2}\left(\frac{2}{x} + \frac{3}{6x-4} - \frac{2}{x+1}\right)$$

5c) $y = x^{2/x}$

$$\ln y = \ln x^{2x^{-1}}$$

$$\ln y = 2x^{-1}\cdot\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 2x^{-1}\cdot\frac{1}{x} + \ln x\cdot -2x^{-2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x^2} - \frac{2\ln x}{x^2}$$

$$\frac{dy}{dx} = y\left(\frac{2 - 2\ln x}{x^2}\right)$$

or

$$\frac{dy}{dx} = x^{\frac{2}{x}}\left(\frac{2 - 2\ln x}{x^2}\right)$$

6) AP MULTIPLE CHOICE EXAMPLES

1) The slope of the line tangent to the graph of $y = \ln\left(x^2\right)$ at $x = e^2$ is

(A) $\frac{1}{e^2}$ (B) $\frac{2}{e^2}$ (C) $\frac{4}{e^2}$ (D) $\frac{1}{e^4}$ (E) $\frac{4}{e^4}$

$y = \ln(x^2)$ $y' = \frac{1}{x^2} \cdot 2x$

$y' = \frac{2}{x}$ so $y'(e^2) = \frac{2}{e^2}$

OR $y = 2\ln x$
$y' = 2 \cdot \frac{1}{x}$
$= 2/x$

2) The slope of the line tangent to the graph of $y = \ln\left(\frac{x}{2}\right)$ at $x = 4$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1 (E) 4

$y = \ln x - \ln 2$
$y' = \frac{1}{x} - 0$
$y'(4) = \frac{1}{4}$

or $y' = \frac{1}{\left(\frac{x}{2}\right)} \cdot \frac{1}{2}$
$y' = \frac{2}{x} \cdot \frac{1}{2}$
$y' = \frac{1}{x}$

3) If $y = \frac{\ln x}{x}$, then $\frac{dy}{dx} =$

(A) $\frac{1}{x}$ (B) $\frac{1}{x^2}$ (C) $\frac{\ln x - 1}{x^2}$ (D) $\frac{1 - \ln x}{x^2}$ (E) $\frac{1 + \ln x}{x^2}$

$y' = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$y' = \frac{1 - \ln x}{x^2}$

4) $\frac{d}{dx}\left(x^{\ln x}\right) =$

(A) $x^{\ln x}$ (B) $(\ln x)^x$ (C) $\frac{2}{x}(\ln x)\left(x^{\ln x}\right)$ (D) $(\ln x)\left(x^{\ln x - 1}\right)$ (E) $2(\ln x)\left(x^{\ln x}\right)$

If $y = x^{\ln x}$

then $\ln y = \ln x^{\ln x}$

$\ln y = \ln x \cdot \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x}$

$\frac{dy}{dx} = y\left(\frac{2\ln x}{x}\right)$

or $x^{\ln x} \cdot \frac{2}{x} \cdot \ln x$

5) The slope of the line <u>normal</u> to the graph of $y = 2\ln(\sec x)$ at $x = \frac{\pi}{4}$ is

(A) -2

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 2

(E) nonexistent

$y' = 2 \cdot \frac{1}{\sec x} \cdot \sec x \tan x$

$y' = 2\tan x$

$y'\left(\frac{\pi}{4}\right) = 2 \cdot \tan\left(\frac{\pi}{4}\right)$

$= 2 \cdot 1$

$= 2$

6) If $f(x) = e^{3\ln(x^2)}$, then $f'(x) =$

(A) $e^{3\ln(x^2)}$ (B) $\frac{3}{x^2}e^{3\ln(x^2)}$ (C) $6(\ln x)e^{3\ln(x^2)}$ (D) $5x^4$ (E) $6x^5$

$f(x) = e^{\ln[(x)^2]^3}$

because of inverses → $f(x) = (x^2)^3$

$f(x) = x^6$

$f'(x) = 6x^5$

or $f'(x) = e^{3 \cdot \ln(x^2)} \cdot 3 \cdot \frac{1}{x^2} \cdot 2x$

$= e^{\ln(x^2)^3} \cdot 3 \cdot \frac{2}{x}$

$= (x^2)^3 \cdot \frac{6}{x}$

$= x^6 \cdot \frac{6}{x}$

$= 6x^5$