AP Calculus Optimization Critical Homework

Find two positive numbers that satisfy the given requirements.

1a) The product is 147 and the sum of the first number plus three times the second number is a minimum.

Let x and y be two positive numbers such that
$xy = 147$.

$$S = x + 3y = \frac{147}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{147}{y^2} = 0 \text{ when } y = 7.$$

$$\frac{d^2S}{dy^2} = \frac{294}{y^3} > 0 \text{ when } y = 7.$$

S is minimum when $y = 7$ and $x = 21$.

1b) The sum of the first number and twice the second number is 108 and the product is a maximum.

$P = a \cdot b$

$S = a + 2b$

$108 = a + 2b$

$108 - 2b = a$

$P = (108 - 2b) \cdot b$

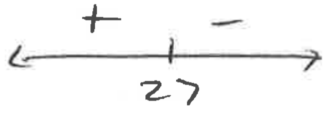
$P = 108b - 2b^2$

$P' = 108 - 4b$

$0 = 108 - 4b$

$\frac{-108}{-4} = b$

$27 = b$



27 maximizes the product if used for "b"

$\boxed{b=27}$

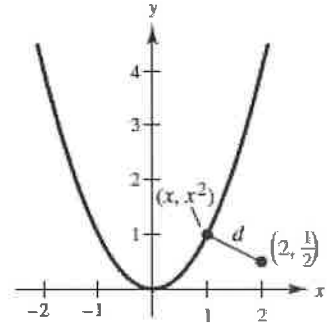
$a(27) = 108 - 2(27)$

$a(27) = 54$

$\boxed{a = 54}$

Find the point on the graph of the function that is closest to the given point.

2a) $f(x)=x^2$ and the point $\left(2,\frac{1}{2}\right)$



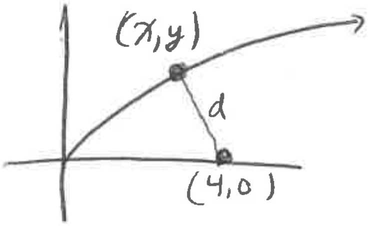
$$d = \sqrt{(x-2)^2 + \left[x^2 - (1/2)\right]^2}$$
$$= \sqrt{x^4 - 4x + (17/4)}$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$
$$f'(x) = 4x^3 - 4 = 0$$
$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.

2b) $f(x)=\sqrt{x}$ and the point $(4,0)$



$$d^2 = (x-4)^2 + (y-0)^2$$
$$d = \sqrt{(x-4)^2 + y^2}$$
$$d = \sqrt{(x-4)^2 + (\sqrt{x})^2}$$
$$d = \sqrt{x^2-8x+16+x}$$
$$d = (x^2-7x+16)^{1/2}$$
$$d' = \frac{1}{2}(x^2-7x+16)^{-1/2}\cdot(2x-7)$$
$$d' = \frac{2x-7}{2\sqrt{x^2-7x+16}}$$
$$0 = \frac{2x-7}{2\sqrt{x^2-7x+16}}$$

$$0 = 2x-7$$
$$\frac{7}{2} = x$$

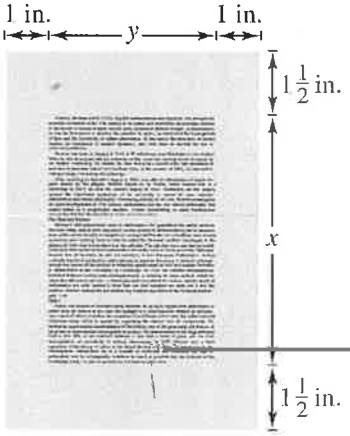
− +

$\frac{7}{2}$

$\frac{7}{2}$ minimizes "d" if $x = 7/2$ is used

$$f\left(\frac{7}{2}\right) = \sqrt{\frac{7}{2}}$$
$$\boxed{\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)}$$

3a) A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



Solution Let A be the area to be minimized.

$$A = (x + 3)(y + 2) \qquad \text{Primary equation}$$

The printed area inside the margins is given by

$$24 = xy. \qquad \text{Secondary equation}$$

Solving this equation for y produces $y = 24/x$. Substitution into the primary equation produces

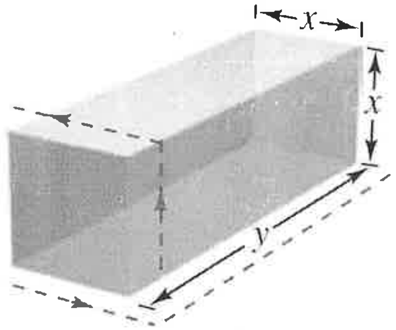
$$A = (x + 3)\left(\frac{24}{x} + 2\right) = 30 + 2x + \frac{72}{x}.$$

Because x must be positive, you are interested only in values of A for $x > 0$. To find the critical numbers, differentiate with respect to x.

$$\frac{dA}{dx} = 2 - \frac{72}{x^2} = 0 \implies x^2 = 36$$

So, the critical numbers are $x = \pm 6$. You do not have to consider $x = -6$ because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x = 6$. So, $y = \frac{24}{6} = 4$ and the dimensions of the page should be $x + 3 = 9$ inches by $y + 2 = 6$ inches.

3b) A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)



$V = x^2 y$

$V = x^2(108 - 4x)$

$V = 108x^2 - 4x^3$

$V' = 216x - 12x^2$

$0 = 216x - 12x^2$

$0 = 12x \cdot (18 - x)$

$12x = 0$ or $18 - x = 0$

$x = 0$ or $18 = x$

$108 = l + P_{cross}$

$108 = y + 4x$

$108 - 4x = y$

$+ \quad -$

$\longleftarrow 18 \longrightarrow$

18 maximizes V when $x = 18$.

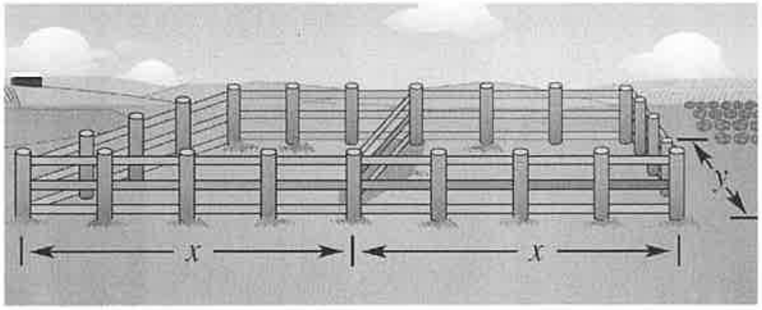
$y(18) = 108 - 4(18)$

$y(18) = 36$

Dimensions

18in x 18in x 36in

3c) A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?



$200 = 4x + 3y$

$\frac{200-4x}{3} = y$

$A = 2x \cdot y$

$A = 2x \cdot \left(\frac{200-4x}{3}\right)$

$A = 2x\left(\frac{200}{3} - \frac{4}{3}x\right)$

$A = \frac{400}{3}x - \frac{8}{3}x^2$

$A' = \frac{400}{3} - \frac{16}{3}x$

$0 = \frac{400}{3} - \frac{16}{3}x$

$0 = 400 - 16x$

$\frac{400}{16} = x$

$25 = x$

$$\xleftarrow{\quad + \quad} 25 \xrightarrow{\quad - \quad}$$

Area is maximized when $\boxed{x = 25}$

$y(25) = \frac{200-4(25)}{3}$

$y(25) = \boxed{100/3}$

Dimensions

25 ft $= x$

$\frac{100}{3}$ ft $= y$

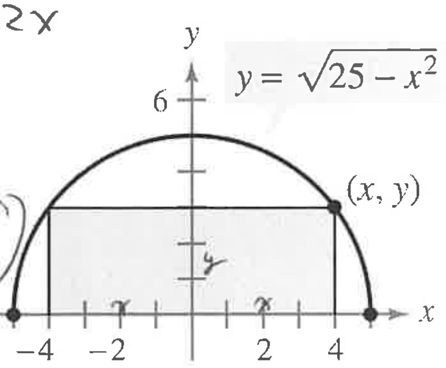
3d) A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?

Since $L = 2x$

$L = \cancel{2}\cdot\frac{5\sqrt{2}}{\cancel{2}}$

$\boxed{5\sqrt{2}}$

$W = \sqrt{\frac{50}{4}}$ o. $\boxed{\frac{5\sqrt{2}}{2}}$



$A = L \cdot W$

$A = \underline{2x} \cdot y$

$A = 2x \cdot (25 - x^2)^{1/2}$

$A' = 2x \cdot \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) + 2\,(25 - x^2)^{1/2}$

$A' = \frac{-2x^2}{\sqrt{25 - x^2}} + 2\sqrt{25 - x^2}$

$\sqrt{25 - x^2} \cdot 0 = \left(\frac{-2x^2}{\sqrt{25 - x^2}} + 2\sqrt{25 - x^2}\right) \cdot \sqrt{25 - x^2}$

$0 = -2x^2 + 2\,(25 - x^2)$

$0 = -2x^2 + 50 - 2x^2$

$4x^2 = 50$

$x^2 = \frac{50}{4}$ $\qquad x = \sqrt{\frac{50}{4}}$ or $\frac{5\sqrt{2}}{2}$

$$\xleftarrow{\quad + \quad} \frac{5\sqrt{2}}{2} \xrightarrow{\quad - \quad}$$

Area maximized when

$x = \frac{5\sqrt{2}}{2}$

$y\left(\sqrt{\frac{50}{4}}\right) = \sqrt{25 - \left(\sqrt{\frac{50}{4}}\right)^2}$

" $= \sqrt{25 - \frac{50}{4}} = \sqrt{\frac{50}{4}}$

4) AP MULTIPLE CHOICE EXAMPLES

1)

If $y = 2x - 8$, what is the minimum value of the product xy ?

(A) −16 [✓ (B) −8] (C) −4 (D) 0 (E) 2

$P = xy$

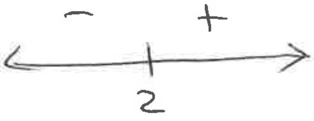
$P = x(2x-8)$

$P = 2x^2 - 8x$

$P' = 4x - 8$

$0 = 4x - 8$

$2 = x$



Product is minimum when $x=2$

$y(2) = 2(2) - 8$

$y(2) = 4 - 8$

$y(2) = -4$

$P = xy$

$P = (2)\cdot(-4)$

$\boxed{P = -8}$

2)

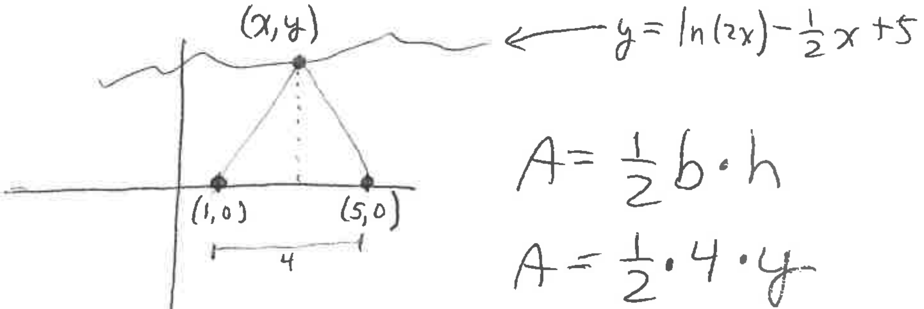
Consider a triangle in the xy-plane. Two vertices of the triangle are on the x-axis at $(1, 0)$ and $(5, 0)$, and a third vertex is on the graph of $y = \ln(2x) - \frac{1}{2}x + 5$ for $\frac{1}{2} \le x \le 8$. What is the maximum area of such a triangle?

(A) $\frac{19}{2}$

(B) $2\ln 2 + 9$

[✓ (C) $2\ln 4 + 8$]

(D) $2\ln 16 + 2$



$A = \frac{1}{2}b\cdot h$

$A = \frac{1}{2}\cdot 4\cdot y$

$A = 2y$

$A = 2\left(\ln(2x) - \frac{1}{2}x + 5\right)$

$A = 2\ln(2x) - x + 10$

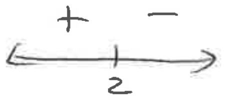
$A' = 2\cdot\frac{1}{2x}\cdot 2 - 1$

$A' = \frac{2}{x} - 1$

$0 = \frac{2}{x} - 1$

$0 = 2 - x$

$x = 2$



Area maximized when $x=2$

$y(2) = \ln 4 - 1 + 5$

" $= \ln 4 + 4$

$A = 2y$

$A = 2(\ln 4 + 4)$

$A = 2\ln 4 + 8$