Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 16$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$

2. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$y' = \frac{x}{y}$

3. $x^{1/2} + y^{1/2} = 9$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$

4. $x^3 + y^3 = 8$

$3x^2 + 3y^2y' = 0$

$y' = -\frac{x^2}{y^2}$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \frac{y - 3x^2}{2y - x}$

6. $x^2y + y^2x = -2$

$x^2y' + 2xy + y^2 + 2yxy' = 0$

$(x^2 + 2xy)y' = -(y^2 + 2xy)$

$y' = \frac{-y(y + 2x)}{x(x + 2y)}$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$

8. $(xy)^{1/2} - x + 2y = 0$

$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$

$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$

$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}\,y' = 0$

$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$

9. $x^3 - 2x^2y + 3xy^2 = 38$

$3x^2 - 2x^2y' - 4xy + 6xyy' + 3y^2 = 0$

$2x(3y - x)y' = 4xy - 3x^2 - 3y^2$

$y' = \frac{4xy - 3x^2 - 3y^2}{2x(3y - x)}$

10. $2 \sin x \cos y = 1$

$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$

$y' = \frac{\cos x \cos y}{\sin x \sin y}$

$= \cot x \cot y$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \frac{\cos x}{4 \sin 2y}$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$

$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$

$y' = \frac{\cos \pi x}{\sin \pi y}$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

15. $y = \sin(xy)$

$$y' = [xy' + y]\cos(xy)$$

$$y' - x\cos(xy)y' = y\cos(xy)$$

$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

16. $x = \sec\frac{1}{y}$

$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

17. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At $(-4, -1)$: $y' = -\frac{1}{4}$

18. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(1, 1)$: $y' = \frac{2}{3}$.

19. $y^2 = \frac{x^2 - 9}{x^2 + 9}$

$$2yy' = \frac{(x^2 + 9)(2x) - (x^2 - 9)2x}{(x^2 + 9)^2}$$

$$y' = \frac{18x}{(x^2 + 9)^2 y}$$

At $(3, 0)$: y' is undefined.

20. $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$.

21. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At $(8, 1)$: $y' = -\frac{1}{2}$.

22. $x^3 + y^3 - 2xy = 0$

$$3x^2 + 3y^2y' - 2xy' - 2y = 0$$

$$(3y^2 - 2x)y' = 2y - 3x^2$$

$$y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

At $(1, 1)$: $y' = -1$.

23. $\tan(x + y) = x$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$.

24. $x \cos y = 1$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x}\cot y = \frac{\cot y}{x}$$

At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$.

25. $\sqrt{x} + \sqrt{y} = 3$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{(1/2)x^{-1/2}}{(1/2)y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

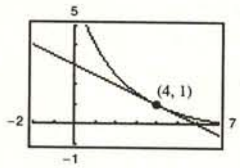
At $(4, 1)$: $y' = -\frac{1}{2}$.

Tangent line:

$$y - 1 = -\frac{1}{2}(x - 4)$$

$$y = -\frac{1}{2}x + 3$$

$$x + 2y - 6 = 0$$



26. $y^2 = \frac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

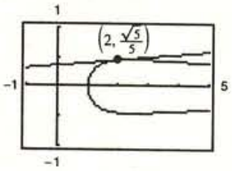
At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{[(2\sqrt{5})/5](4 + 1)^2} = \frac{1}{10\sqrt{5}}$.

Tangent line:

$$y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



27. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

28. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$.

29.

$$(x^2 + y^2)^2 = 4x^2y$$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At $(1, 1)$: $y' = 0$.

30.

$$x^3 + y^3 - 6xy = 0$$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

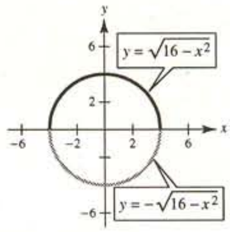
At $\left(\frac{4}{3}, \frac{8}{3}\right)$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$.

31. (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm\sqrt{16 - x^2}} = \frac{-x}{y}$$

(d) Implicitly:

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

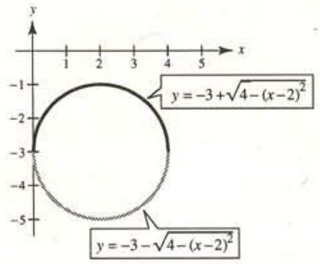
32. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}[4 - (x - 2)^2]^{-1/2}(-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{\pm\sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$

(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$(2y + 6)y' = -2(x - 2)$$

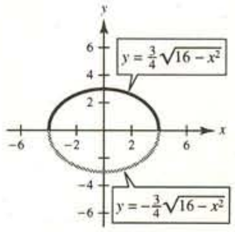
$$y' = \frac{-(x - 2)}{y + 3}$$

33. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm\frac{3}{4}\sqrt{16 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{3}{8}(16 - x^2)^{-1/2}(-2x)$$

$$= \mp\frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y}$$

(d) Implicitly:

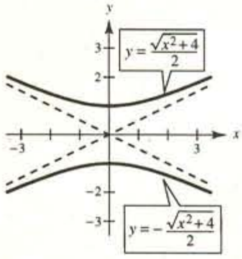
$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

34. (a) $y^2 = 1 + \frac{x^2}{4} = \frac{x^2 + 4}{4}$

$$y = \pm\frac{1}{2}\sqrt{x^2 + 4}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{4}(x^2 + 4)^{-1/2}(2x)$$

$$= \frac{\pm x}{2\sqrt{x^2 + 4}}$$

$$= \frac{\pm x}{4(1/2)\sqrt{x^2 + 4}} = \frac{x}{4y}$$

(d) Implicitly:

$$2yy' - \frac{1}{2}x = 0$$

$$y' = \frac{x}{4y}$$

35.

$$x^2 + xy = 5$$

$$2x + xy' + y = 0$$

$$y' = \frac{-(2x + y)}{x}$$

$$2 + xy'' + y' + y' = 0$$

$$xy'' = -2(1 + y')$$

$$y'' = \frac{-2[1 - (2x + y)/x]}{x} = \frac{2(x + y)}{x^2}$$

$$y'' = \frac{2(x + y)}{x^2} \cdot \frac{x}{x} = \frac{10}{x^3}$$

(**Note:** You could write $y = (5 - x^2)/x$ and calculate y' and y'' directly.)

36.

$$x^2y^2 - 2x = 3$$
$$2x^2yy' + 2xy^2 - 2 = 0$$
$$x^2yy' + xy^2 - 1 = 0$$
$$y' = \frac{1 - xy^2}{x^2y}$$
$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$
$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$
$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$
$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$
$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$
$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

37.

$$x^2 - y^2 = 16$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$
$$x - yy' = 0$$
$$1 - yy'' - (y')^2 = 0$$
$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$
$$y^2 - y^3y'' = x^2$$
$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

38.

$$1 - xy = x - y$$
$$y - xy = x - 1$$
$$y = \frac{x - 1}{1 - x} = -1$$
$$y' = 0$$
$$y'' = 0$$

39.

$$y^2 = x^3$$
$$2yy' = 3x^2$$
$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$
$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$
$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$
$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

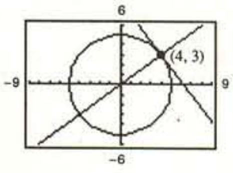
40.

$$y^2 = 4x$$
$$2yy' = 4$$
$$y' = \frac{2}{y}$$
$$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2}\right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

41. $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

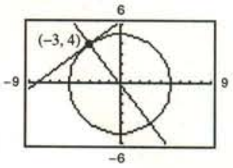
At $(4, 3)$:



Tangent line: $y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0.$

At $(-3, 4)$:



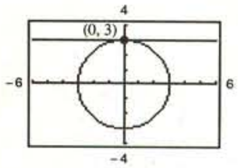
Tangent line: $y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0.$

42. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

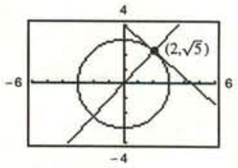
At $(0, 3)$:



Tangent line: $y = 3$

Normal line: $x = 0.$

At $(2, \sqrt{5})$:



Tangent line: $y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Rightarrow 2x + \sqrt{5}y - 9 = 0$

Normal line: $y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Rightarrow \sqrt{5}x - 2y = 0.$

43. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

44. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y} = 1 \text{ at } (1, 2)$$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $\left(1 + 2\sqrt{2}, 2 - 2\sqrt{2}\right)$ and $\left(1 - 2\sqrt{2}, 2 + 2\sqrt{2}\right)$

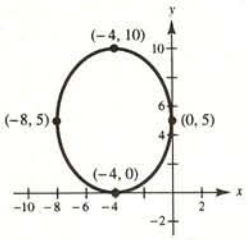
Equations: $\left(x - 1 - 2\sqrt{2}\right)^2 + \left(y - 2 + 2\sqrt{2}\right)^2 = 16$

$$\left(x - 1 + 2\sqrt{2}\right)^2 + \left(y - 2 - 2\sqrt{2}\right)^2 = 16$$

45. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$



Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

Horizontal tangents: $(-4, 0), (-4, 10)$.

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

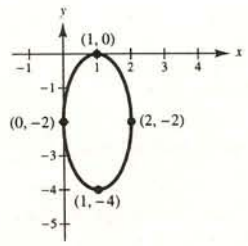
$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents: $(0, 5), (-8, 5)$.

46. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

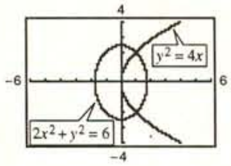
$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$.

47. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.



<u>Ellipse:</u>	<u>Parabola:</u>
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\frac{2x}{y}$	$y' = \frac{2}{y}$

At (1, 2), the slopes are:

$y' = -1$ $\qquad$ $y' = 1.$

At (1, −2), the slopes are:

$y' = 1$ $\qquad$ $y' = -1.$

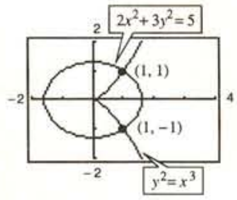
Tangents are perpendicular.

48. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$



<u>$y^2 = x^3$:</u>	<u>$2x^2 + 3y^2 = 5$:</u>
$2yy' = 3x^2$	$4x + 6yy' = 0$
$y' = \frac{3x^2}{2y}$	$y' = -\frac{2x}{3y}$

At (1, 1), the slopes are:

$y' = \frac{3}{2}$ $\qquad$ $y' = -\frac{2}{3}.$

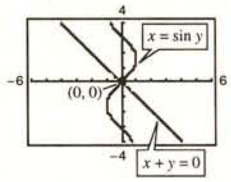
At (1, −1), the slopes are:

$y' = -\frac{3}{2}$ $\qquad$ $y' = \frac{2}{3}.$

Tangents are perpendicular.

49. $y = -x$ and $x = \sin y$

Point of intersection: (0, 0)



<u>$y = -x$:</u>	<u>$x = \sin y$:</u>
$y' = -1$	$1 = y' \cos y$
	$y' = \sec y$

At (0, 0), the slopes are:

$y' = -1$ $\qquad$ $y' = 1.$

Tangents are perpendicular.