Section 5.9 Inverse Trigonometric Functions and Integration

1. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1-(3x)^2}}(3)\,dx = \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$$

2. $$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$$

3. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

4. $$\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\arctan\frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$$

5. $$\int \frac{1}{x\sqrt{4x^2-1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2-1}}\,dx = \operatorname{arcsec}|2x| + C$$

6. $$\int \frac{1}{4+(x-1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

7. $$\int \frac{x^3}{x^2+1}\,dx = \int\left[x - \frac{x}{x^2+1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$$ (Use long division.)

8. $$\int \frac{x^4-1}{x^2+1}\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + C$$

9. $$\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \arcsin(x+1) + C$$

10. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4+16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2+(t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

11. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1-t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1-(t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

12. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4-4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2-2^2}}(2x)\,dx$$
$$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

13. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

14. Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}}\,dx$$
$$= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

15. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^0 (1-x^2)^{-1/2}(-2x)\,dx$$
$$= \left[-\sqrt{1-x^2}\right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2}$$
$$\approx -0.134$$

16. Let $u = 1 + x^2$, $du = 2x\,dx$.

$$\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1+x^2}(2x)\,dx$$
$$= \left[\frac{1}{2}\ln(1+x^2)\right]_{-\sqrt{3}}^{0} = -\ln 2$$

17. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

18. $$\int_1^2 \frac{1}{3+(x-2)^2}\,dx = \int_1^2 \frac{1}{(\sqrt{3})^2+(x-2)^2}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right)\right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

19. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$$
$$= \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

20. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int \frac{1}{1+u^2}\,du = 2\arctan(u) + C$$
$$= 2\arctan(\sqrt{x}) + C$$

21. $$\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2 \frac{1}{1+(x-1)^2}\,dx = \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

22. $$\int_{-3}^{-1} \frac{1}{x^2+6x+13}\,dx = \int_{-3}^{-1} \frac{1}{(x+3)^2+4}\,dx = \left[\frac{1}{2}\arctan\frac{x+3}{2}\right]_{-3}^{-1} = \frac{\pi}{8}$$

23. $$\int \frac{2x}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}dx - 6\int \frac{1}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{4+(x+3)^2}\,dx$$
$$= \ln|x^2+6x+13| - 3\arctan\left(\frac{x+3}{2}\right) + C$$

24. $$\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - 7\int \frac{1}{1+(x+1)^2}\,dx = \ln|x^2+2x+2| - 7\arctan(x+1) + C$$

25. $$\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$$

26. $$\int \frac{1}{\sqrt{-x^2+2x}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1) + C$$

27. Let $u = -x^2 - 4x$, $du = (-2x-4)\,dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx = -\frac{1}{2}\int (-x^2-4x)^{-1/2}(-2x-4)\,dx$$
$$= -\sqrt{-x^2-4x} + C$$

28. Let $u = x^2 - 2x$, $du = (2x-2)\,dx$.

$$\int \frac{x-1}{\sqrt{x^2-2x}}\,dx = \frac{1}{2}\int (x^2-2x)^{-1/2}(2x-2)\,dx$$
$$= \sqrt{x^2-2x} + C$$

29. $$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx = -\int_2^3 (4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$
$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

30. $$\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx = \operatorname{arcsec}|x-1| + C$$

31. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^2 + 1}\,dx$$
$$= \frac{1}{2}\arctan(x^2 + 1) + C$$

32. Let $u = x^2 - 4$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}}\,dx$$
$$= \frac{1}{2}\arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

33. (a) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C,\ u = x$

(b) $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\,dx = -\sqrt{1 - x^2} + C,\ u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1 - x^2}}\,dx$ cannot be evaluated using the basic integration rules.

34. (a) $\displaystyle\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C,\ u = x^2$

(c) $\displaystyle\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C,\ u = \frac{1}{x}$

35. (a) $\displaystyle\int \sqrt{x - 1}\,dx = \frac{2}{3}(x - 1)^{3/2} + C,\ u = x - 1$

(b) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x - 1}\,dx = \int (u^2 + 1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x - 1)^{3/2}[3(x - 1) + 5] + C = \frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$$

(c) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x - 1}}\,dx = \int \frac{u^2 + 1}{u}(2u)\,du = 2\int (u^2 + 1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2 + 3) + C = \frac{2}{3}\sqrt{x - 1}(x + 2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

36. (a) $\displaystyle\int \frac{1}{1 + x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{2x}{1 + (x^2)^2}\,dx = \frac{1}{2}\arctan(x^2) + C,\ u = x^2$

(c) $\displaystyle\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1 + x^4}\,dx = \frac{1}{4}\ln(1 + x^4) + C,\ u = 1 + x^4$

37. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2 + 3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2 + 3}\,du = \int 2\,du - \int 6\frac{1}{u^2 + 3}\,du$$
$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

38. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$
$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$