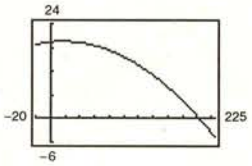
43. (a) $V = 2\pi \int_0^{200} xf(x)\,dx$

$$\approx \frac{2\pi(200)}{3(8)}[0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$\approx 1{,}366{,}593$ cubic feet

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x)\,dx \approx 2\pi(213{,}800) = 1{,}343{,}345$ cubic feet

(d) Number gallons $\approx V(7.48) = 10{,}048{,}221$ gallons

Section 6.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2}\,dx = \left[\frac{13}{5}x\right]_0^5 = 13$$

2. $(1, 2), (7, 10)$

(a) $d = \sqrt{(7-1)^2 + (10-2)^2} = 10$

(b) $y = \frac{4}{3}x + \frac{2}{3}$

$y' = \frac{4}{3}$

$$s = \int_1^7 \sqrt{1 + \left(\frac{4}{3}\right)^2}\,dx = \left[\frac{5}{3}x\right]_1^7 = 10$$

3. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, [0, 1]$

$$s = \int_0^1 \sqrt{1 + x}\,dx$$

$$= \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left(\sqrt{8} - 1\right) \approx 1.219$$

4. $y = x^{3/2} - 1$

$y' = \frac{3}{2}x^{1/2}, [0, 4]$

$$s = \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx$$

$$= \frac{4}{9}\int_0^4 \left(1 + \frac{9}{4}x\right)^{1/2}\left(\frac{9}{4}\right) dx$$

$$= \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^4$$

$$= \frac{8}{27}(10^{3/2} - 1) \approx 9.073$$

5. $y = \frac{3}{2}x^{2/3}$

$$y' = \frac{1}{x^{1/3}}, [1, 8]$$

$$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2}\, dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\, dx$$

$$= \frac{3}{2}\int_1^8 \sqrt{x^{2/3} + 1}\left(\frac{2}{3x^{1/3}}\right) dx$$

$$= \frac{3}{2}\left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

6. $y = \frac{x^5}{10} + \frac{1}{6x^3}$

$$y' = \frac{1}{2}x^4 - \frac{1}{2x^4}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2, [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{2x^4}\right) dx$$

$$= \left[\frac{1}{10}x^5 - \frac{1}{6x^3}\right]_1^2 = \frac{779}{240} \approx 3.246$$

7. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}$$

$$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$$

8. $y = \frac{1}{2}(e^x + e^{-x})$

$$y' = \frac{1}{2}(e^x - e^{-x})$$

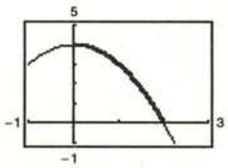
$$1 + (y')^2 = \left[\frac{1}{2}(e^x + e^{-x})\right]^2, [0, 2]$$

$$s = \int_0^2 \sqrt{\left[\frac{1}{2}(e^x + e^{-x})\right]^2}\, dx$$

$$= \frac{1}{2}\int_0^2 (e^x + e^{-x})\, dx$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) \approx 3.627$$

9. (a) $y = 4 - x^2, 0 \le x \le 2$



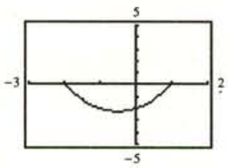
(b) $y' = -2x$

$$1 + (y')^2 = 1 + 4x^2$$

$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx$$

(c) $L \approx 4.647$

10. (a) $y = x^2 + x - 2, -2 \le x \le 1$



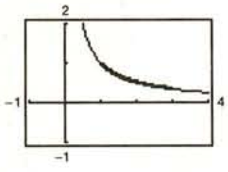
(b) $y' = 2x + 1$

$$1 + (y')^2 = 1 + 4x^2 + 4x + 1$$

$$L = \int_{-2}^1 \sqrt{2 + 4x + 4x^2}\, dx$$

(c) $L \approx 5.653$

11. (a) $y = \frac{1}{x}, 1 \le x \le 3$



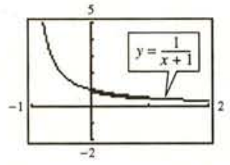
(b) $$y' = -\frac{1}{x^2}$$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(c) $L \approx 2.147$

12. (a) $y = \frac{1}{1+x}, 0 \le x \le 1$



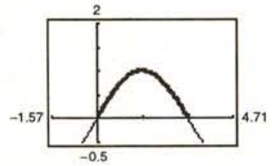
(b) $$y' = -\frac{1}{(1+x)^2}$$

$$1 + (y')^2 = 1 + \frac{1}{(1+x)^4}$$

$$L = \int_0^1 \sqrt{1 + \frac{1}{(1+x)^4}}\, dx$$

(c) $L \approx 1.132$

13. (a) $y = \sin x, 0 \le x \le \pi$



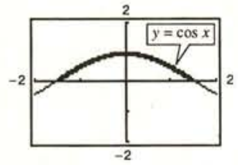
(b) $$y' = \cos x$$

$$1 + (y')^2 = 1 + \cos^2 x$$

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

(c) $L \approx 3.820$

14. (a) $y = \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



(b) $$y' = -\sin x$$

$$1 + (y')^2 = 1 + \sin^2 x$$

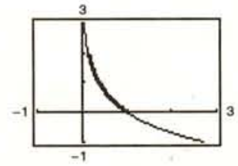
$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

(c) 3.820

15. (a) $x = e^{-y}, 0 \le y \le 2$

$y = -\ln x$

$1 \ge x \ge e^{-2} \approx 0.135$



(b) $$y' = -\frac{1}{x}$$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^{1} \sqrt{1 + \frac{1}{x^2}}\, dx$$

(c) $L \approx 2.221$

Alternatively, you can do all the computations with respect to y.

(a) $x = e^{-y}\ 0 \le y \le 2$

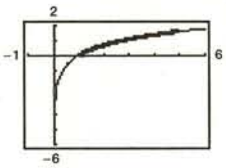
(b) $$\frac{dx}{dy} = -e^{-y}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}}\, dy$$

(c) $L \approx 2.221$

16. (a) $y = \ln x,\ 1 \le x \le 5$



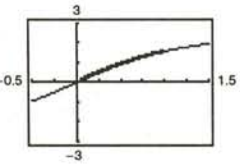
(b) $y' = \dfrac{1}{x}$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_1^5 \sqrt{1 + \frac{1}{x^2}}\,dx$$

(c) $L \approx 4.367$

17. (a) $y = 2 \arctan x,\ 0 \le x \le 1$



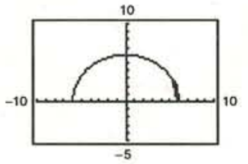
(b) $y' = \dfrac{2}{1 + x^2}$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1 + x^2)^2}}\,dx$$

(c) $L \approx 1.871$

18. (a) $x = \sqrt{36 - y^2},\ 0 \le y \le 3$

$y = \sqrt{36 - x^2},\ 3\sqrt{3} \le x \le 6$



(b) $\dfrac{dx}{dy} = \dfrac{1}{2}(36 - y^2)^{-1/2}(-2y)$

$$= \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \frac{y^2}{36 - y^2}}\,dy$$

$$= \int_0^3 \frac{6}{\sqrt{36 - y^2}}\,dy$$

(c) $L \approx 3.142$ (π!)

Alternatively, you can convert to a function of x.

$$y = \sqrt{36 - x^2}$$

$$y' = \frac{dy}{dx} = -\frac{x}{\sqrt{36 - x^2}}$$

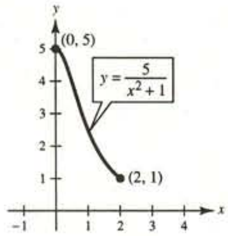
$$L = \int_{3\sqrt{3}}^6 \sqrt{1 + \frac{x^2}{36 - x^2}}\,dx = \int_{3\sqrt{3}}^6 \frac{6}{\sqrt{36 - x^2}}\,dx$$

Although this integral is undefined at $x = 0$, a graphing utility still gives $L \approx 3.142$.

19. $\displaystyle\int_0^2 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{5}{x^2 + 1}\right)\right]^2}\,dx$

$s \approx 5$

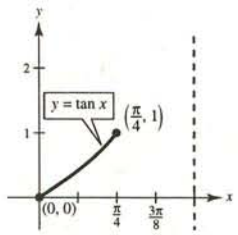
Matches (b)



20. $\displaystyle\int_0^{\pi/4} \sqrt{1 + \left[\frac{d}{dx}(\tan x)\right]^2}\,dx$

$s \approx 1$

Matches (e)



31. $y = \frac{x^3}{3}$

$y' = x^2, [0, 3]$

$$S = 2\pi \int_0^3 \frac{x^3}{3}\sqrt{1 + x^4}\,dx$$

$$= \frac{\pi}{6}\int_0^3 (1 + x^4)^{1/2}(4x^3)\,dx$$

$$= \left[\frac{\pi}{9}(1 + x^4)^{3/2}\right]_0^3$$

$$= \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$$

32. $y = \sqrt{x}$

$y' = \frac{1}{2\sqrt{x}}, [1, 4]$

$$S = 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\,dx$$

$$= 2\pi \int_1^4 \frac{\sqrt{x}}{2\sqrt{x}}\sqrt{4x + 1}\,dx$$

$$= \pi \int_1^4 \sqrt{4x + 1}\,dx$$

$$= \frac{\pi}{4}\int_1^4 (4x + 1)^{1/2}(4)\,dx$$

$$= \left[\frac{\pi}{6}(4x + 1)^{3/2}\right]_1^4$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \approx 30.85$$

33. $y = \frac{x^3}{6} + \frac{1}{2x}$

$y' = \frac{x^2}{2} - \frac{1}{2x^2}$

$1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2, [1, 2]$

$$S = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right)dx$$

$$= 2\pi\left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 = \frac{47\pi}{16}$$

34. $y = \frac{x}{2}$

$y' = \frac{1}{2}$

$1 + (y')^2 = \frac{5}{4}, [0, 6]$

$$S = 2\pi \int_0^6 \frac{x}{2}\sqrt{\frac{5}{4}}\,dx$$

$$= \left[\frac{2\pi\sqrt{5}}{8}x^2\right]_0^6 = 9\sqrt{5}\,\pi$$

35. $y = \sqrt[3]{x} + 2$

$y' = \frac{1}{3x^{2/3}}, [1, 8]$

$$S = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3} + 1}\,dx$$

$$= \frac{\pi}{18}\int_1^8 (9x^{4/3} + 1)^{1/2}(12x^{1/3})\,dx$$

$$= \left[\frac{\pi}{27}(9x^{4/3} + 1)^{3/2}\right]_1^8$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$$

36. $y = 4 - x^2$

$y' = -2x, [0, 2]$

$$S = 2\pi \int_0^2 x\sqrt{1 + 4x^2}\,dx$$

$$= \frac{\pi}{4}\int_0^2 (1 + 4x^2)^{1/2}(8x)\,dx$$

$$= \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^2$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.18$$

37. $y = \sin x$

$$y' = \cos x, [0, \pi]$$

$$S = 2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\, dx$$

$$\approx 14.4236$$

38. $y = \ln x$

$$y' = \frac{1}{x}$$

$$1 + (y')^2 = \frac{x^2 + 1}{x^2}, [1, e]$$

$$S = 2\pi \int_1^e x\sqrt{\frac{x^2 + 1}{x^2}}\, dx$$

$$= 2\pi \int_1^e \sqrt{x^2 + 1}\, dx \approx 22.943$$

39. $y = \dfrac{hx}{r}$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$S = 2\pi \int_0^r x\sqrt{\frac{r^2 + h^2}{r^2}}\, dx$$

$$= \left[\frac{2\pi\sqrt{r^2 + h^2}}{r}\left(\frac{x^2}{2}\right)\right]_0^r = \pi r\sqrt{r^2 + h^2}$$

40. $y = \sqrt{r^2 - x^2}$

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (y')^2 = \frac{r^2}{r^2 - x^2}$$

$$S = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\, dx$$

$$= 2\pi \int_{-r}^{r} r\, dx = \Big[2\pi r x\Big]_{-r}^{r} = 4\pi r^2$$

41. $y = \sqrt{9 - x^2}$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}}\, dx$$

$$= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}}\, dx$$

$$= \Big[-6\pi\sqrt{9 - x^2}\Big]_0^2$$

$$= 6\pi\left(3 - \sqrt{5}\right) \approx 14.40$$

See figure in Exercise 42.

42. From Exercise 41 we have:

$$S = 2\pi \int_0^a \frac{rx}{\sqrt{r^2 - x^2}}\, dx$$

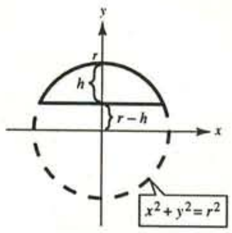
$$= -r\pi \int_0^a \frac{-2x\, dx}{\sqrt{r^2 - x^2}}$$

$$= \Big[-2r\pi\sqrt{r^2 - x^2}\Big]_0^a$$

$$= 2r^2\pi - 2r\pi\sqrt{r^2 - a^2}$$

$$= 2r\pi\left(r - \sqrt{r^2 - a^2}\right)$$

$= 2\pi r h$ (where h is the height of the zone)



43. $y = \dfrac{1}{3}x^{1/2} - x^{3/2}$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$S = 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{\frac{1}{36}(x^{-1/2} + 9^{1/2})^2}\, dx = \frac{2\pi}{6}\int_0^{1/3}\left(\frac{1}{3}x^{1/2} - x^{3/2}\right)(x^{-1/2} + 9x^{1/2})\, dx$$

$$= \frac{\pi}{3}\int_0^{1/3}\left(\frac{1}{3} + 2x - 9x^2\right) dx = \frac{\pi}{3}\left[\frac{1}{3}x + x^2 - 3x^3\right]_0^{1/3} = \frac{\pi}{27}\text{ ft}^2 \approx 0.1164\text{ ft}^2 \approx 16.8\text{ in}^2$$

Amount of glass needed: $V = \dfrac{\pi}{27}\left(\dfrac{0.015}{12}\right) \approx 0.00015\text{ ft}^3 \approx 0.25\text{ in}^3$