CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Exercises

1. $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

2. $f(x) = \cos\dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2}\sin\dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

3. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f'(4) = 0$

4. $f(x) = -3x\sqrt{x + 1}$

$f'(x) = -3x\left[\dfrac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}(-3)$

$= -\dfrac{3}{2}(x + 1)^{-1/2}[x + 2(x + 1)]$

$= -\dfrac{3}{2}(x + 1)^{-1/2}(3x + 2)$

$f'\left(-\dfrac{2}{3}\right) = 0$

5. $f(x) = (x + 2)^{2/3}$

$f'(x) = \dfrac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

6. Using the limit definition of the derivative,

$\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{(4 - |x|) - 4}{x} = 1$

$\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{(4 - |x|) - 4}{x - 0} = -1$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

7. $f(x) = x^2(x - 3) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, x = 2$

8. $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

9. $g(t) = t\sqrt{4 - t}$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}[-t + 2(4 - t)]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical numbers: $t = 4, t = \frac{8}{3}$

10. $f(x) = \dfrac{4x}{x^2 + 1}$

$f'(x) = \dfrac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \dfrac{4(1 - x^2)}{(x^2 + 1)^2}$

Critical numbers: $x = \pm 1$

11. $h(x) = \sin^2 x + \cos x, 0 \le x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

Critical numbers: $x = 0, x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

12. $f(\theta) = 2 \sec \theta + \tan \theta, 0 \le \theta < 2\pi$

$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$

$= \sec \theta(2 \tan \theta + \sec \theta)$

$= \sec \theta\left[2\left(\frac{\sin \theta}{\cos \theta}\right) + \frac{1}{\cos \theta}\right]$

$= \sec^2 \theta(2 \sin \theta + 1)$

Critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

13. $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2 \Longrightarrow$ No critical numbers

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

14. $f(x) = \frac{2x + 5}{3}, [0, 5]$

$f'(x) = \frac{2}{3} \Longrightarrow$ No critical numbers

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

15. $f(x) = -x^2 + 3x, [0, 3]$

$f'(x) = -2x + 3$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

16. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

17. $f(x) = x^3 - 3x^2, [-1, 3]$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Left endpoint: $(-1, -4)$ Minimum

Critical number: $(0, 0)$ Maximum

Critical number: $(2, -4)$ Minimum

Right endpoint: $(3, 0)$ Maximum

18. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

19. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

20. $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \frac{1}{3x^{2/3}}$

Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

21. $h(t) = 4 - |t - 4|, [1, 6]$

From the graph of the function on the interval [1, 6] you can determine the following.

Left endpoint: (1, 1) Minimum

Critical number: (4, 4) Maximum

Right endpoint: (6, 2)

22. $g(t) = \dfrac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \dfrac{6t}{(t^2 + 3)^2}$

Left endpoint: $\left(-1, \dfrac{1}{4}\right)$ Maximum

Critical number: (0, 0) Minimum

Right endpoint: $\left(1, \dfrac{1}{4}\right)$ Maximum

23. $h(s) = \dfrac{1}{s - 2}, [0, 1]$

$h'(s) = \dfrac{-1}{(s - 2)^2}$

Left endpoint: $\left(0, -\dfrac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

24. $h(t) = \dfrac{t}{t - 2}, [3, 5]$

$h'(t) = \dfrac{-2}{(t - 2)^2}$

Left endpoint: (3, 3) Maximum

Right endpoint: $\left(5, \dfrac{5}{3}\right)$ Minimum

25. $f(x) = \cos \pi x, \left[0, \dfrac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: (0, 1) Maximum

Right endpoint: $\left(\dfrac{1}{6}, \dfrac{\sqrt{3}}{2}\right)$ Minimum

26. $g(x) = \csc x, \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = -\csc x \cot x$

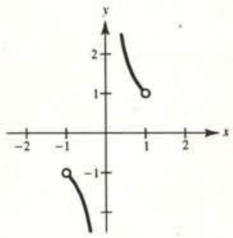
Left endpoint: $\left(\dfrac{\pi}{6}, 2\right)$ Maximum

Right endpoint: $\left(\dfrac{\pi}{3}, \dfrac{2\sqrt{3}}{3}\right)$ Minimum

27. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \to \pi/2^-} \tan x = \infty$.

28. Let $f(x) = 1/x$. f is continuous on (0, 1) but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



29. (a) Yes

(b) No

30. (a) No

(b) Yes

31. (a) No

(b) Yes

32. (a) No

(b) Yes

33. (a) Minimum: $(0, -3)$

Maximum: (2, 1)

(b) Minimum: $(0, -3)$

(c) Maximum: (2, 1)

(d) No extrema

34. (a) Minimum: (4, 1)

Maximum: (1, 4)

(b) Maximum: (1, 4)

(c) Minimum: (4, 1)

(d) No extrema

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

36. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

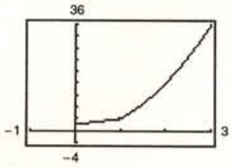
(c) Maximum: $(0, 2)$

(d) Maximum: $(1, \sqrt{3})$

37. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

Left endpoint: $(0, 2)$ Minimum

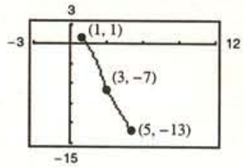
Right endpoint: $(3, 36)$ Maximum



38. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

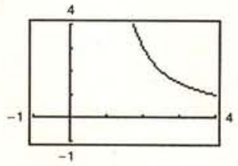
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum



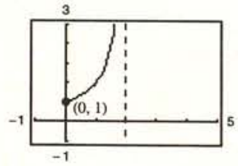
39. $f(x) = \dfrac{3}{x - 1}, (1, 4]$

Right endpoint: $(4, 1)$ Minimum

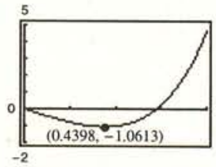


40. $f(x) = \dfrac{2}{2 - x}, [0, 2)$

Left endpoint: $(0, 1)$ Minimum



41. (a)



Maximum: $(1, 4.7)$

Minimum: $(0.4398, -1.0613)$

(b)

$$f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: $(0.4398, -1.0613)$