69. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

70. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f.

71. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Longrightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Longrightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2,\ y'' = 2$

Concave upward: $(-\infty, \infty)$

2. $y = -x^3 + 3x^2 - 2,\ y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

3. $f(x) = \dfrac{24}{x^2 + 12},\ y'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

4. $f(x) = \dfrac{x^2 - 1}{2x + 1},\ y'' = \dfrac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

6. $y = \dfrac{1}{270}(-3x^5 + 40x^3 + 135x),\ y'' = \dfrac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

7. $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

$f''(x) = -2$

Critical number: $x = 3$

$f''(3) < 0$

Therefore, $(3, 9)$ is relative maximum.

8. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

9. $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

10. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

11. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

12. $f(x) = 5 + 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

$f''(x) = 6 - 6x = 6(1 - x)$

Critical numbers: $x = 0, x = 2$

$f''(0) = 6 > 0$

Therefore, $(0, 5)$ is a relative minimum.

$f''(2) = -6 < 0$

Therefore, $(2, 9)$ is a relative maximum.

13. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

14. $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

15. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

16. $f(x) = \sqrt{x^2 + 1}$

$$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Critical number: $x = 0$

$$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

17. $f(x) = x + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$

$f''(x) = \frac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

18. $f(x) = \frac{x}{x - 1}$

$f'(x) = \frac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

19. $f(x) = \cos x - x,\ 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

20. $f(x) = 2 \sin x + \cos 2x, 0 \le x \le 2\pi$

$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2 \sin x - 4 \cos 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$f''\left(\frac{\pi}{2}\right) > 0$

$f''\left(\frac{5\pi}{6}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

21. $f(x) = x^3 - 12x$

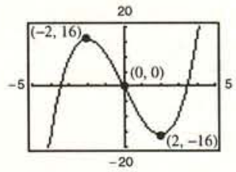
$f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2) = 0$ when $x = \pm 2$.

$f''(x) = 6x$

$f''(-2) = -12 < 0 \Rightarrow (-2, 16)$ is a relative maximum.

$f''(2) = 12 > 0 \Rightarrow (2, -16)$ is a relative minimum.

$f''(x) = 6x = 0$ when $x = 0$.



Test interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

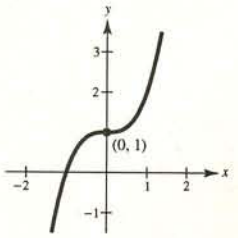
Point of inflection: $(0, 0)$

22. $f(x) = x^3 + 1$

$f'(x) = 3x^2 = 0$ when $x = 0$.

$f''(x) = 6x = 0$ when $x = 0$.

Since $f'(x) \geq 0$ for all x and the concavity changes at $x = 0$. $(0, 1)$ is a point of inflection.



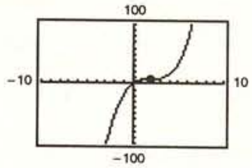
23. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$= 3(x - 2)^2 = 0$ when $x = 2$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

Since $f'(x) > 0$ when $x \neq 2$ and the concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.



24. $f(x) = 2x^3 - 3x^2 - 12x$

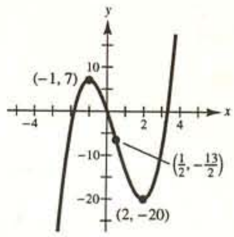
$f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1) = 0$ when $x = -1, 2$.

$f''(x) = 12x - 6$

$f''(-1) = -18 < 0 \Longrightarrow (-1, 7)$ is a relative maximum.

$f''(2) = 18 > 0 \Longrightarrow (2, -20)$ is a relative minimum.

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.



Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

25. $f(x) = \frac{1}{4}x^4 - 2x^2$

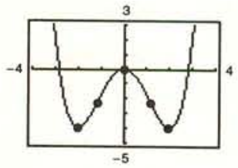
$f'(x) = x^3 - 4x = x(x + 2)(x - 2) = 0$ when $x = 0, \pm 2$.

$f''(x) = 3x^2 - 4$

$f''(-2) = 8 > 0 \Longrightarrow (-2, -4)$ is a relative minimum.

$f''(0) = -4 < 0 \Longrightarrow (0, 0)$ is a relative maximum.

$f''(2) = 8 > 0 \Longrightarrow (2, -4)$ is a relative minimum.



$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\frac{2}{\sqrt{3}}$.

Test interval	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

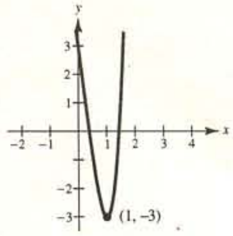
Points of inflection: $\left(\pm\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

26. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8 = 8(x - 1)(x^2 + x + 1) = 0$ when $x = 1$.

$f''(x) = 24x^2 = 0$ when $x = 0$.

Since $f''(1) > 0$, $(1, -3)$ is a relative minimum. However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x.



27. $f(x) = x(x - 4)^3$

$f'(x) = x[3(x - 4)^2] + (x - 4)^3$

$= (x - 4)^2(4x - 4) = 4(x - 1)(x - 4)^2 = 0$ when $x = 1, 4$.

$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$

$= 4(x - 4)[2(x - 1) + (x - 4)]$

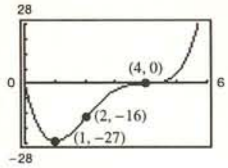
$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$

$f''(1) = 36 > 0 \Rightarrow (1, -27)$ is a relative minimum.

$f''(4) = 0 \Rightarrow$ test fails

By the First Derivative Test we see that $x = 4$ does not yield a relative extrema.

$f''(x) = 12(x - 4)(x - 2) = 0$ when $x = 2, 4$.



Test interval	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16)$, $(4, 0)$

28. $f(x) = x^3(x - 4)$

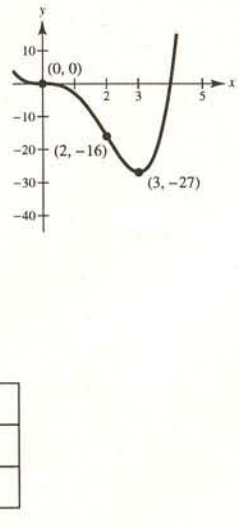
$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3) = 0$ when $x = 0, 3$.

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(0) = 0 \Rightarrow$ test fails

$f''(3) = 36 > 0 \Rightarrow (3, -27)$ is a relative minimum.



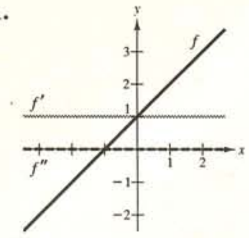
By the First Derivative Test we see that $x = 0$ does not yield a relative extrema.

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

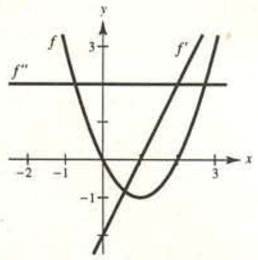
Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0)$, $(2, -16)$

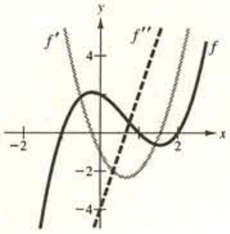
41.



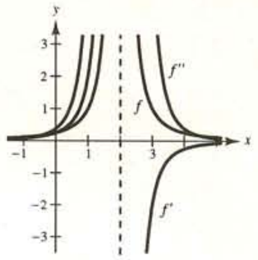
42.



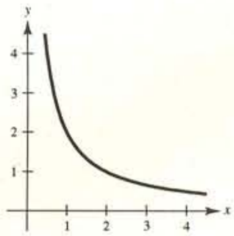
43.



44.



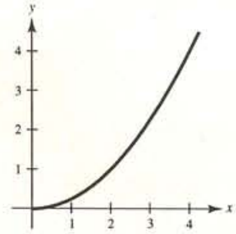
45. (a)



$f' < 0$ means f decreasing

f' increasing means concave upward

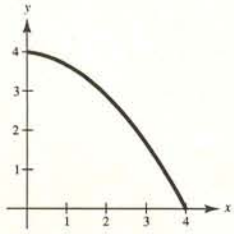
(b)



$f' > 0$ means f increasing

f' increasing means concave upward

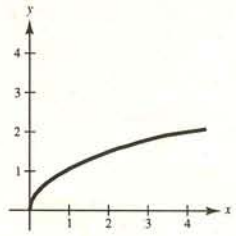
46. (a)



$f' < 0$ means f decreasing

f' decreasing means concave downwards

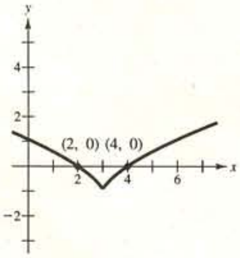
(b)



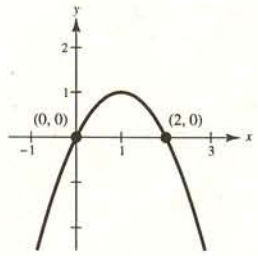
$f' > 0$ means f increasing

f' decreasing means concave downward

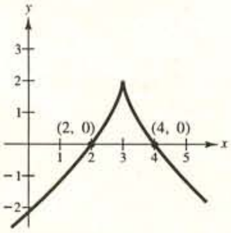
47.



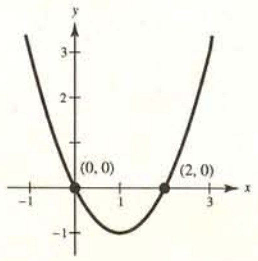
48.



49.



50.



77. True. Let $y = ax^3 + bx^2 + cx + d, a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

78. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

79. False

$$f(x) = 3 \sin x + 2 \cos x$$
$$f'(x) = 3 \cos x - 2 \sin x$$
$$3 \cos x - 2 \sin x = 0$$
$$3 \cos x = 2 \sin x$$
$$\frac{3}{2} = \tan x$$

Critical number: $x = \tan^{-1}\left(\frac{3}{2}\right)$

$f\left(\tan^{-1} \frac{3}{2}\right) \approx 3.60555$ is the maximum value of y.

80. True

$$y = \sin(bx)$$

Slope: $y' = b \cos(bx)$

$-b \leq y' \leq b$ (Assume $b > 0$)

Section 3.5 Limits at Infinity

1. $f(x) = \dfrac{3x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 3$

Matches (f)

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

3. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

Matches (d)

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

5. $f(x) = \dfrac{4\sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptotes: $y = 0$

Matches (b)

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

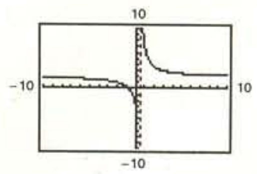
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e)

7. $f(x) = \dfrac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2



$\lim_{x \to \infty} f(x) = 2$