Section 5.2 The Natural Logarithmic Function and Integration

1. $u = x + 1, du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

2. $u = x - 5, du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

3. $u = 3 - 2x, du = -2\,dx$

$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\int \frac{1}{3-2x}(-2)\,dx$$
$$= -\frac{1}{2}\ln|3-2x| + C$$

4. $u = 6x + 1, du = 6\,dx$

$$\int \frac{1}{6x+1}\,dx = \frac{1}{6}\int \frac{1}{6x+1}(6)\,dx$$
$$= \frac{1}{6}\ln|6x+1| + C$$

5. $u = x^2 + 1, du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x^2+1}(2x)\,dx$$
$$= \frac{1}{2}\ln(x^2+1) + C$$
$$= \ln\sqrt{x^2+1} + C$$

6. $u = 3 - x^3, du = -3x^2\,dx$

$$\int \frac{x^2}{3-x^3}\,dx = -\frac{1}{3}\int \frac{1}{3-x^3}(-3x^2)\,dx$$
$$= -\frac{1}{3}\ln|3-x^3| + C$$

7. $$\int \frac{x^2-4}{x}\,dx = \int \left(x - \frac{4}{x}\right)dx$$
$$= \frac{x^2}{2} - 4\ln|x| + C$$

8. $u = 9 - x^2, du = -2x\,dx$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx$$
$$= -\sqrt{9-x^2} + C$$

9. $u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx$$
$$= \frac{1}{3}\ln|x^3+3x^2+9x| + C$$

10. $u = x^2 + 6x + 7, du = 2(x+3)\,dx$

$$\int \frac{x+3}{x^2+6x+7}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+6x+7}\,dx$$
$$= \frac{1}{2}\ln|x^2+6x+7| + C$$

11. $u = \ln x, du = \frac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$$

12. $u = \ln x, du = \frac{1}{x}\,dx$

$$\int \frac{1}{x\ln x^2}\,dx = \frac{1}{2}\int \frac{1}{x\ln x}\,dx$$
$$= \frac{1}{2}\ln|\ln|x|| + C$$

13. $u = x + 1, du = dx$

$$\int \frac{1}{\sqrt{x+1}}\,dx = \int (x+1)^{-1/2}\,dx$$
$$= 2(x+1)^{1/2} + C$$
$$= 2\sqrt{x+1} + C$$

14. $u = 1 + x^{1/3}, du = \frac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx = 3\int \frac{1}{1+x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx$$
$$= 3\ln|1+x^{1/3}| + C$$

15. $u = \sqrt{x} - 3,\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2(u + 3)\,du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx = 2\int \frac{(u+3)^2}{u}\,du = 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int \left(u + 6 + \frac{9}{u}\right)du$$

$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 = u^2 + 12u + 18\ln|u| + C_1$$

$$= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1$$

$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C \text{ where } C = C_1 - 27.$$

16. $u = 1 + \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx \Rightarrow (u - 1)\,du = dx$

$$\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int \left(u - \frac{1}{u}\right)du$$

$$= u - \ln|u| + C_1$$

$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$

$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

17. $$\int \frac{2x}{(x-1)^2}\,dx = \int \frac{2x - 2 + 2}{(x-1)^2}\,dx$$

$$= \int \frac{2(x-1)}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$

$$= 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$

$$= 2\ln|x - 1| - \frac{2}{(x-1)} + C$$

18. $$\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3}\,dx$$

$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x - 1| + \frac{1}{2(x-1)^2} + C$$

19. $$\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$$

$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

20. $$\int \tan 5\theta\,d\theta = \frac{1}{5}\int \frac{5\sin 5\theta}{\cos 5\theta}\,d\theta$$

$$= -\frac{1}{5}\ln|\cos 5\theta| + C$$

21. $$\int \csc 2x\,dx = \frac{1}{2}\int (\csc 2x)(2)\,dx$$

$$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$$

22. $$\int \sec\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\right)dx$$

$$= 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$$

23. $$\int \frac{\cos t}{1 + \sin t}\,dt = \ln|1 + \sin t| + C$$

24. $$\int \frac{\sin x}{1 + \cos x}\,dx = -\int \frac{-\sin x}{1 + \cos x}\,dx$$

$$= -\ln|1 + \cos x| + C$$

25. $$\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$$

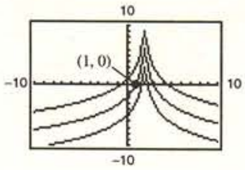
26. $$\int (\sec t + \tan t)\,dt = \ln|\sec t + \tan t| - \ln|\cos t| + C$$

$$= \ln\left|\frac{\sec t + \tan t}{\cos t}\right| + C = \ln|\sec t(\sec t + \tan t)| + C$$

27. $y = \int \frac{3}{2 - x}\,dx$

$= -3\int \frac{1}{x - 2}\,dx$

$= -3\ln|x - 2| + C$



$(1, 0)$: $0 = -3\ln|1 - 2| + C \Rightarrow C = 0$

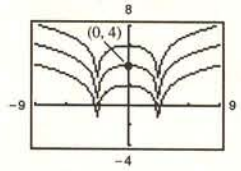
$y = -3\ln|x - 2|$

28. $y = \int \frac{2x}{x^2 - 9}\,dx$

$= \ln|x^2 - 9| + C$

$(0, 4)$: $4 = \ln|0 - 9| + C \Rightarrow C = 4 - \ln 9$

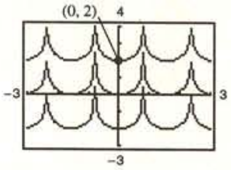
$y = \ln|x^2 - 9| + 4 - \ln 9$



29. $s = \int \tan(2\theta)\,d\theta$

$= \frac{1}{2}\int \tan(2\theta)(2\,d\theta)$

$= -\frac{1}{2}\ln|\cos 2\theta| + C$



$(0, 2)$: $2 = -\frac{1}{2}\ln|\cos(0)| + C \Rightarrow C = 2$

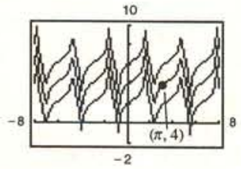
$s = -\frac{1}{2}\ln|\cos 2\theta| + 2$

30. $r = \int \frac{\sec^2 t}{\tan t + 1}\,dt$

$= \ln|\tan t + 1| + C$

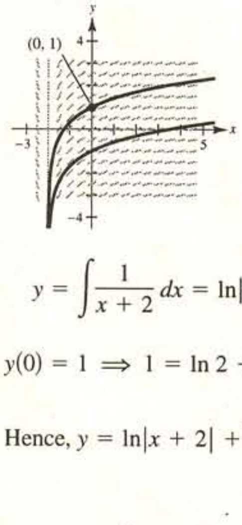
$(\pi, 4)$: $4 = \ln|0 + 1| + C \Rightarrow C = 4$

$r = \ln|\tan t + 1| + 4$



31. $\frac{dy}{dx} = \frac{1}{x + 2}$, $(0, 1)$

(a)



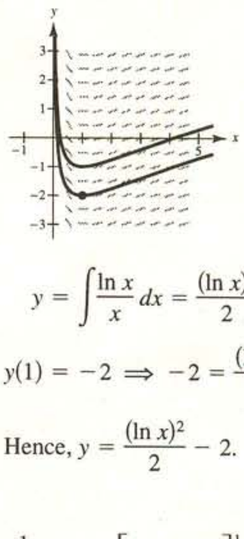
(b) $y = \int \frac{1}{x + 2}\,dx = \ln|x + 2| + C$

$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

Hence, $y = \ln|x + 2| + 1 - \ln 2 = \ln\left|\frac{x + 2}{2}\right| + 1$.

32. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$

(a)



(b) $y = \int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2$

Hence, $y = \frac{(\ln x)^2}{2} - 2$.

33. $\int_0^4 \frac{5}{3x + 1}\,dx = \left[\frac{5}{3}\ln|3x + 1|\right]_0^4$

$= \frac{5}{3}\ln 13 \approx 4.275$

34. $\int_{-1}^{1} \frac{1}{x + 2}\,dx = \left[\ln|x + 2|\right]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

35. $u = 1 + \ln x,\ du = \frac{1}{x}\,dx$

$$\int_1^e \frac{(1 + \ln x)^2}{x}\,dx = \left[\frac{1}{3}(1 + \ln x)^3\right]_1^e = \frac{7}{3}$$

36. $u = \ln x,\ du = \frac{1}{x}\,dx$

$$\int_e^{e^2} \frac{1}{x \ln x}\,dx = \int_e^{e^2} \left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_e^{e^2} = \ln 2$$

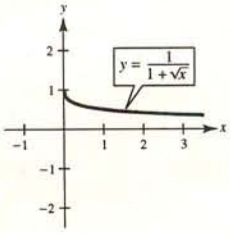
37. $$\int_0^2 \frac{x^2 - 2}{x + 1}\,dx = \int_0^2 \left(x - 1 - \frac{1}{x + 1}\right)dx = \left[\frac{1}{2}x^2 - x - \ln|x + 1|\right]_0^2 = -\ln 3$$

38. $$\int_0^1 \frac{x - 1}{x + 1}\,dx = \int_0^1 1\,dx + \int_0^1 \frac{-2}{x + 1}\,dx = \Big[x - 2\ln|x + 1|\Big]_0^1 = 1 - 2\ln 2$$

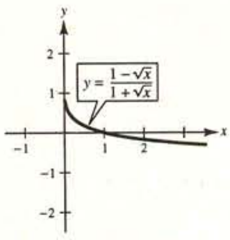
39. $$\int_1^2 \frac{1 - \cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2 = \ln\left|\frac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$$

40. $$\begin{aligned}\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta &= \int_{0.1}^{0.2} (\csc^2 2\theta - 2\csc 2\theta \cot 2\theta + \cot^2 2\theta)\,d\theta \\ &= \int_{0.1}^{0.2} (2\csc^2 2\theta - 2\csc 2\theta \cot 2\theta - 1)\,d\theta \\ &= \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024\end{aligned}$$

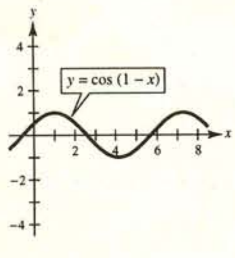
41. $$\begin{aligned}\int \frac{1}{1 + \sqrt{x}}\,dx &= 2(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + C_1 \\ &= 2[\sqrt{x} - \ln(1 + \sqrt{x})] + C \text{ where } C = C_1 + 2.\end{aligned}$$



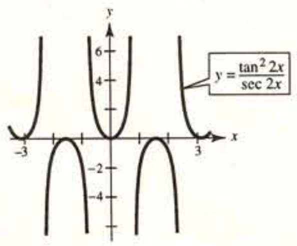
42. $$\begin{aligned}\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}}\,dx &= -(1 + \sqrt{x})^2 + 6(1 + \sqrt{x}) - 4\ln(1 + \sqrt{x}) + C_1 \\ &= 4\sqrt{x} - x - 4\ln(1 + \sqrt{x}) + C \text{ where } C = C_1 + 5.\end{aligned}$$



43. $$\int \cos(1 - x)\,dx = -\sin(1 - x) + C$$



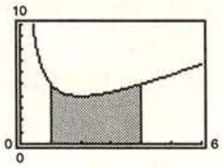
44. $$\int \frac{\tan^2 2x}{\sec 2x}\,dx = \frac{1}{2}[\ln|\sec 2x + \tan 2x| - \sin 2x] + C$$



57. $A = \int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right) dx$

$= \left[\frac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \frac{1}{2}$

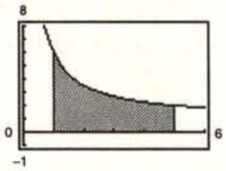
$= \frac{15}{2} + 8\ln 2 \approx 13.045$ square units



58. $A = \int_1^5 \frac{x+5}{x}\,dx = \int_1^5 \left(1 + \frac{5}{x}\right) dx$

$= \left[x + 5\ln x\right]_1^5$

$= 4 + 5\ln 5 \approx 12.047$ square units

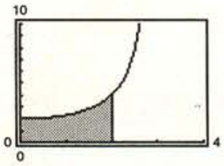


59. $\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$

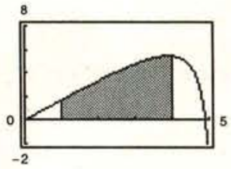
$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1 + 0|$

$= \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$



60. $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right]_1^4$

$= \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



61. $P(t) = \int \frac{3000}{1+0.25t}\,dt = (3000)(4)\int \frac{0.25}{1+0.25t}\,dt = 12{,}000\ln|1 + 0.25t| + C$

$P(0) = 12{,}000\ln|1 + 0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1 + 0.25t| + 1000 = 1000[12\ln|1 + 0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

62. $t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\,dT$

$= \frac{10}{\ln 2}\left[\ln(T - 100)\right]_{250}^{300} = \frac{10}{\ln 2}[\ln 200 - \ln 150] = \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504$ units of time

63. $\frac{1}{50 - 40}\int_{40}^{50} \frac{90{,}000}{400 + 3x}\,dx = \left[3000\ln|400 + 3x|\right]_{40}^{50} \approx \$168.27$