106. (a) $\lim_{x\to0}\frac{1-\cos x}{x^2} = \lim_{x\to0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$= \lim_{x\to0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$

$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$

(b) Thus, $\frac{1-\cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2}x^2$

$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2$ for $x \approx 0$.

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

107. Two functions agree at all but one point means that the functions are identical for all x in their domain, except possibly at one value of x.

Section 1.4 Continuity and One-Sided Limits

1. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

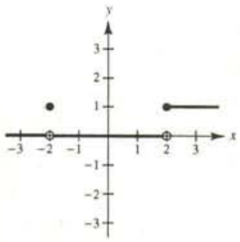
(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

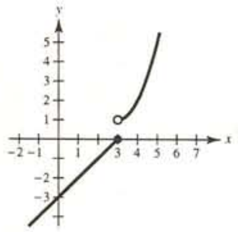
2. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$.

(a) $f(x) = \frac{|x-2|}{x-2}$ (b) $f(x) = \frac{\sin(x+2)}{x+2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \ge 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



3.



The function is not continuous at $x = 3$ because $\lim_{x\to3^+} f(x) = 1 \ne 0 = \lim_{x\to3^-} f(x)$.

4. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm1$.

5. (a) $\lim_{x\to3^+} f(x) = 1$

(b) $\lim_{x\to3^-} f(x) = 1$

(c) $\lim_{x\to3} f(x) = 1$

6. (a) $\lim_{x\to-2^+} f(x) = -2$

(b) $\lim_{x\to-2^-} f(x) = -2$

(c) $\lim_{x\to-2} f(x) = -2$

7. (a) $\lim_{x\to3^+} f(x) = 0$

(b) $\lim_{x\to3^-} f(x) = 0$

(c) $\lim_{x\to3} f(x) = 0$

8. (a) $\lim_{x\to-2^+} f(x) = 2$

(b) $\lim_{x\to-2^-} f(x) = 2$

(c) $\lim_{x\to-2} f(x) = 2$

9. (a) $\lim_{x\to3^+} f(x) = 3$

(b) $\lim_{x\to3^-} f(x) = -3$

(c) $\lim_{x\to3} f(x)$ does not exist.

10. (a) $\lim_{x\to-1^+} f(x) = 0$

(b) $\lim_{x\to-1^-} f(x) = 2$

(c) $\lim_{x\to-1} f(x)$ does not exist.

11. $\lim_{x\to5^+} \frac{x-5}{x^2-25} = \lim_{x\to5^+} \frac{1}{x+5} = \frac{1}{10}$

12. $\lim_{x\to2^+} \frac{2-x}{x^2-4} = \lim_{x\to2^+} -\frac{1}{x+2} = -\frac{1}{4}$

13. $\lim_{x\to2^+} \frac{x}{\sqrt{x^2-4}}$ does not exist since $\frac{x}{\sqrt{x^2-4}}$ grows without bound as $x \to 2^+$.

14.
$$\lim_{x\to4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$$
$$= \lim_{x\to4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x\to4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

15. $\lim_{x\to0} \frac{|x|}{x}$ does not exist since $\lim_{x\to0^+} \frac{|x|}{x} = 1$ and

$\lim_{x\to0^-} \frac{|x|}{x} = -1.$

16.
$$\lim_{x\to2^+} \frac{|x-2|}{x-2} = \lim_{x\to2^+} \frac{x-2}{x-2} = 1$$
$$\lim_{x\to2^-} \frac{|x-2|}{x-2} = \lim_{x\to2^-} \frac{-(x-2)}{x-2} = -1$$
$$\lim_{x\to2^+} \frac{|x-2|}{x-2} \neq \lim_{x\to2^-} \frac{|x-2|}{x-2}$$

Thus, the limit does not exist.

17.
$$\lim_{\Delta x\to0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

18.
$$\lim_{\Delta x\to0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$$
$$= \lim_{\Delta x\to0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to0^+} (2x + \Delta x + 1)$$
$$= 2x + 0 + 1 = 2x + 1$$

19. $\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \frac{12-2x}{3} = 2$

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \frac{x+2}{2} = \frac{5}{2}$

$\lim_{x\to3} f(x)$ does not exist.

20. $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to2} f(x) = 2$

21. $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x+1) = 2$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^3+1) = 2$

$\lim_{x\to1} f(x) = 2$

22. $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (1-x) = 0$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x) = 1$

$\lim_{x\to1} f(x)$ does not exist.

23. $\lim_{x\to\pi} \cot x$ does not exist since

$\lim_{x\to\pi^+} \cot x$ and $\lim_{x\to\pi^-} \cot x$ do not exist.

24. $\lim_{x\to\pi/2} \sec x$ does not exist since

$\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

25. $\lim_{x\to 3^-} (2[\![x]\!] - 1) = 2(2) - 1 = 3$

$([\![x]\!] = 2$ for $2 < x < 3)$

26. $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

27. $f(x) = \dfrac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

28. $f(x) = \dfrac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

29. $f(x) = \dfrac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$.

30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has discontinuity at $x = 1$ since $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$.

31. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

32. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

33. $f(x) = x + \sin x$ is continuous for all real x.

34. $f(x) = \cos \dfrac{\pi x}{2}$ is continuous for all real x.

35. $f(x) = \dfrac{1}{x - 1}$ has a nonremovable discontinuity at $x = 1$ since $\lim_{x\to 1} f(x)$ does not exist.

36. $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$ do not exist.

37. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

38. $f(x) = \dfrac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x + 3} = \frac{1}{6}.$$

39. $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x\to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

40. $f(x) = \dfrac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x + 2} = \frac{1}{3}.$$

41. $f(x) = \dfrac{|x + 2|}{x + 2}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist.

42. $f(x) = \dfrac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x\to 3} f(x)$ does not exist.

43. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

44. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (-2x + 3) = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

45. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (3 - x) = 1 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

46. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-2x) = -4 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

47. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim\limits_{x\to 1} f(x) = 2$ $\qquad \lim\limits_{x\to 5} f(x) = 2$

3. $f(1) = \lim\limits_{x\to 1} f(x)$ $\qquad f(5) = \lim\limits_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

48. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases} = \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim\limits_{x\to -1} f(x) = -1$ $\qquad \lim\limits_{x\to 1} f(x) = 1$

3. $f(-1) = \lim\limits_{x\to -1} f(x)$ $\qquad f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

49. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

50. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

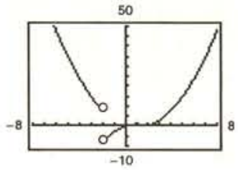
51. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

52. $f(x) = x - [\![x]\!]$ has nonremovable discontinuities at each integer k.

53. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

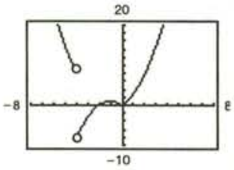
f is not continuous at $x = -2$.



54. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -4$



55. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \dfrac{8}{2^2} = 2$.

56. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (a - 2x) = a$

Let $a = 4$.

57. Find a and b such that $\lim_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ \underline{(+)\,3a + b} &\underline{= -2} \\ 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

58. $\lim_{x\to a} g(x) = \lim_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim_{x\to a} (x + a) = 2a$

Find a such that $2a = 8 \Rightarrow a = 4$.

59. $f(g(x)) = (x - 1)^2$

Continuous for all real x.

60. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

61. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

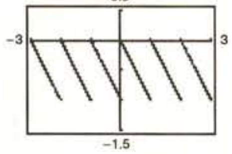
Nonremovable discontinuities at $x = \pm 1$

62. $f(g(x)) = \sin x^2$

Continuous for all real x

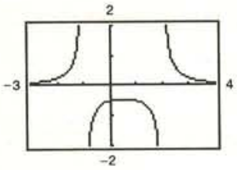
63. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



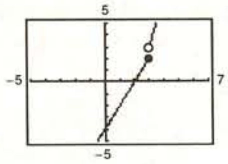
64. $h(x) = \dfrac{1}{(x+1)(x-2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.

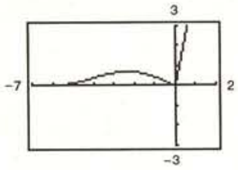


65. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



66. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$



$f(0) = 5(0) = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (5x) = 0$

Therefore, $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

67. $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

68. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty]$

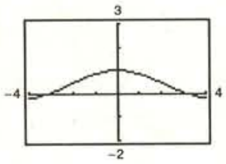
69. $f(x) = \csc \dfrac{x}{2}$

Continuous on: $\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

70. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

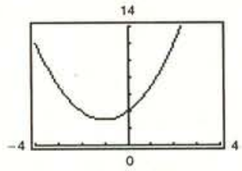
Continuous on $(0, \infty)$

71. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

72. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

73. $f(x) = x^2 - 4x + 3$

$f(x)$ is continuous on $[2, 4]$.

$f(2) = -1$ and $f(4) = 3$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 2 and 4.

74. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

75. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

76. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

77. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

78. $h(0) = 1 + \theta - 3\tan\theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0$.

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

79. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$$-1 < 11 < 29$$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \ (x = -4 \text{ is not in the interval.})$$

Thus, $f(3) = 11$.

80. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$$-1 < 0 < 8$$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$
$$c = 2 \ (x = 4 \text{ is not in the interval.})$$

Thus, $f(2) = 0$.

81. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$$-2 < 4 < 19$$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$
$$(x^2 + x + 3 \text{ has no real solution.})$$
$$c = 2$$

Thus, $f(2) = 4$.

82. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$$f\left(\frac{5}{2}\right) = \frac{35}{6} \text{ and } f(4) = \frac{20}{3}$$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$
$$c = 3 \ (x = 2 \text{ is not in the interval.})$$

Thus, $f(3) = 6$.