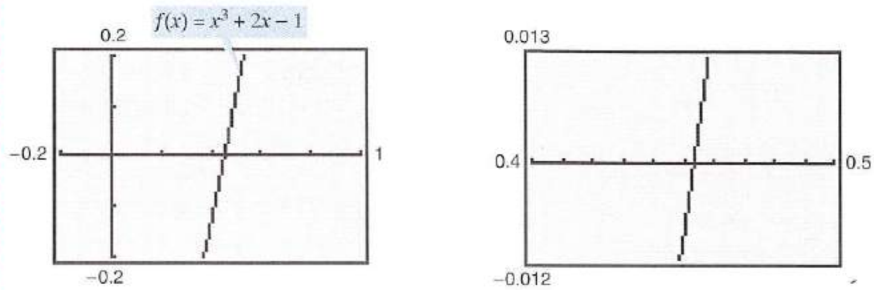
The **bisection method** for approximating the real zeros of a continuous function is similar to the method used in Example 8. If you know that a zero exists in the closed interval $[a, b]$, the zero must lie in the interval

$$\left[a, \frac{a+b}{2}\right] \quad \text{or} \quad \left[\frac{a+b}{2}, b\right].$$

From the sign of $f([a + b]/2)$, you can determine which interval contains the zero. By repeatedly bisecting the interval, you can "close in" on the zero of the function.

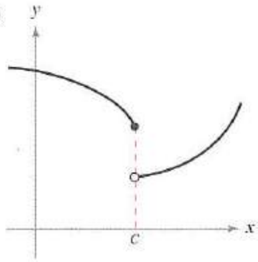
TECHNOLOGY You can also use the zoom feature of a graphing utility to approximate the real zeros of a continuous function. By repeatedly zooming in on the point where the graph crosses the x-axis, and adjusting the x-axis scale, you can approximate the zero of the function to any desired accuracy. The zero of $x^3 + 2x - 1$ is approximately 0.453, as shown in Figure 1.36.

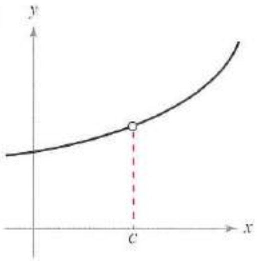


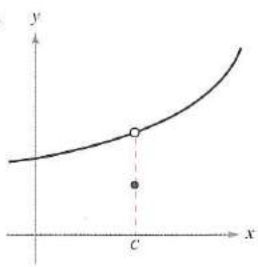
Zooming in on the zero of f
Figure 1.36

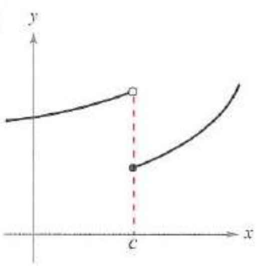
EXERCISES FOR SECTION 1.4

1. ***Think About It*** State how continuity is destroyed at $x = c$ for each of the following.

(a) 

(b) 

(c) 

(d) 

2. ***Writing*** Describe the difference between a discontinuity that is removable and one that is nonremovable. In your explanation, give examples of the following.
   (a) A function with a nonremovable discontinuity at $x = 2$.
   (b) A function with a removable discontinuity at $x = -2$.
   (c) A function that has both of the characteristics described in parts (a) and (b).

3. ***Think About It*** Sketch the graph of any function f such that

$$\lim_{x \to 3^+} f(x) = 1$$

and

$$\lim_{x \to 3^-} f(x) = 0.$$

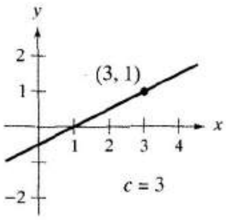
Is the function continuous at $x = 3$? Explain.

4. ***Think About It*** If the functions f and g are continuous for all real x, is $f + g$ always continuous for all real x? Is f/g always continuous for all real x? If either is not continuous, give an example to verify your conclusion.

In Exercises 5–10, use the graph to determine the limit.

(a) $\lim_{x \to c^+} f(x)$ (b) $\lim_{x \to c^-} f(x)$ (c) $\lim_{x \to c} f(x)$

5.



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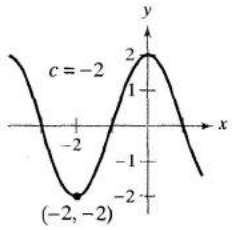

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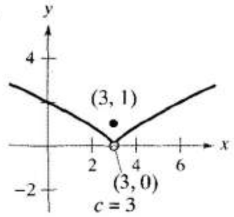

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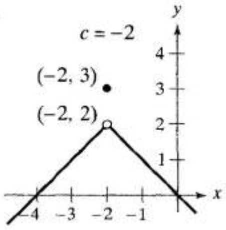

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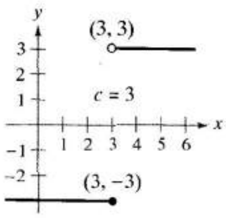

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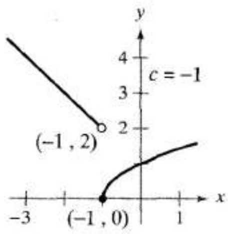


In Exercises, 11–26, find the limit (if it exists).

11. $\lim_{x \to 5^+} \frac{x - 5}{x^2 - 25}$

12. $\lim_{x \to 2^+} \frac{2 - x}{x^2 - 4}$

13. $\lim_{x \to 2^+} \frac{x}{\sqrt{x^2 - 4}}$

14. $\lim_{x \to 4^-} \frac{\sqrt{x} - 2}{x - 4}$

15. $\lim_{x \to 0} \frac{|x|}{x}$

16. $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$

17. $\lim_{\Delta x \to 0^-} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$

18. $\lim_{\Delta x \to 0^+} \frac{(x + \Delta x)^2 + x + \Delta x - (x^2 + x)}{\Delta x}$

19. $\lim_{x \to 3} f(x)$, where $f(x) = \begin{cases} \frac{x + 2}{2}, & x \le 3 \\ \frac{12 - 2x}{3}, & x > 3 \end{cases}$

20. $\lim_{x \to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

21. $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

22. $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

23. $\lim_{x \to \pi} \cot x$

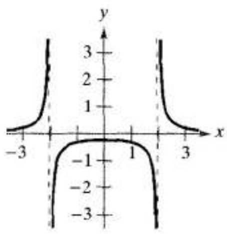
24. $\lim_{x \to \pi/2} \sec x$

25. $\lim_{x \to 3^-} (2[\![x]\!] - 1)$

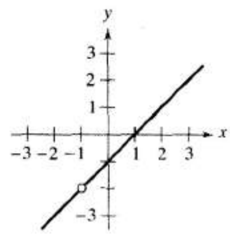
26. $\lim_{x \to 2^+} (2x - [\![x]\!])$

In Exercises 27–30, find the *x*-values (if any) at which *f* is not continuous.

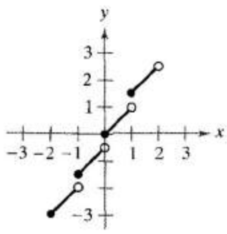
27. $f(x) = \frac{1}{x^2 - 4}$



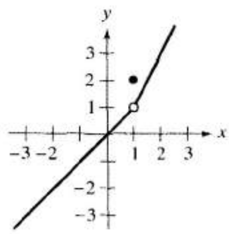
28. $f(x) = \frac{x^2 - 1}{x + 1}$



29. $f(x) = \frac{1}{2}[\![x]\!] + x$



30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



In Exercises 31–52, find the *x*-values (if any) at which *f* is not continuous. Which of the discontinuities are removable?

31. $f(x) = x^2 - 2x + 1$

32. $f(x) = \frac{1}{x^2 + 1}$

33. $f(x) = x + \sin x$

34. $f(x) = \cos \frac{\pi x}{2}$

35. $f(x) = \frac{1}{x - 1}$

36. $f(x) = \frac{x}{x^2 - 1}$

37. $f(x) = \frac{x}{x^2 + 1}$

38. $f(x) = \frac{x - 3}{x^2 - 9}$

39. $f(x) = \frac{x + 2}{x^2 - 3x - 10}$

40. $f(x) = \frac{x - 1}{x^2 + x - 2}$

41. $f(x) = \frac{|x + 2|}{x + 2}$

42. $f(x) = \frac{|x - 3|}{x - 3}$

43. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

44. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

45. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

46. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

47. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

48. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

49. $f(x) = \csc 2x$

50. $f(x) = \tan \dfrac{\pi x}{2}$

51. $f(x) = [\![x - 1]\!]$

52. $f(x) = x - [\![x]\!]$

In Exercises 53 and 54, use a graphing utility to graph the function. From the graph, estimate

$$\lim_{x\to 0^+} f(x) \quad \text{and} \quad \lim_{x\to 0^-} f(x).$$

Is the function continuous on the entire real line?

53. $f(x) = \dfrac{|x^2 - 4|x}{x + 2}$

54. $f(x) = \dfrac{|x^2 + 4x|(x + 2)}{x + 4}$

In Exercises 55–58, find the constants a and b such that the function is continuous on the entire real line.

55. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$

56. $g(x) = \begin{cases} \dfrac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$

57. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

58. $g(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

In Exercises 59–62, discuss the continuity of the composite function $h(x) = f(g(x))$.

59. $f(x) = x^2$
$g(x) = x - 1$

60. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = x - 1$

61. $f(x) = \dfrac{1}{(x - 6)}$
$g(x) = x^2 + 5$

62. $f(x) = \sin x$
$g(x) = x^2$

In Exercises 63–66, use a graphing utility to graph the function. Use the graph to determine any x-values at which the function is not continuous.

63. $f(x) = [\![x]\!] - x$

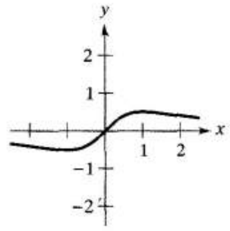
64. $h(x) = \dfrac{1}{x^2 - x - 2}$

65. $g(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

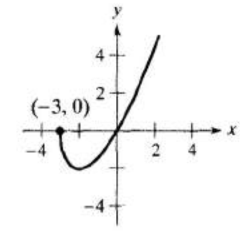
66. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

In Exercises 67–70, find the interval(s) on which the function is continuous.

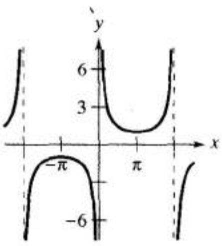
67. $f(x) = \dfrac{x}{x^2 + 1}$



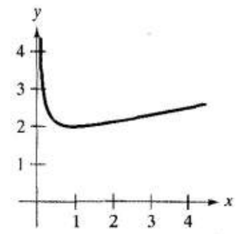
68. $f(x) = x\sqrt{x + 3}$



69. $f(x) = \csc \dfrac{x}{2}$



70. $f(x) = \dfrac{x + 1}{\sqrt{x}}$



Writing **In Exercises 71 and 72, use a graphing utility to graph the function on the interval $[-4, 4]$. Does the graph of the function *appear* continuous on this interval? *Is* the function continuous on $[-4, 4]$? Write a short paragraph about the importance of examining a function analytically as well as graphically.**

71. $f(x) = \dfrac{\sin x}{x}$

72. $f(x) = \dfrac{x^3 - 8}{x - 2}$

Writing **In Exercises 73 and 74, give a written explanation of why the function has a zero in the specified interval.**

73. $f(x) = x^2 - 4x + 3, \quad [2, 4]$

74. $f(x) = x^3 + 3x - 2, \quad [0, 1]$

In Exercises 75–78, use the Intermediate Value Theorem and a graphing utility to approximate the zero of the function in the interval $[0, 1]$. Repeatedly "zoom in" on the graph of the function to approximate the zero accurate to two decimal places. Use the root-finding capabilities of the graphing utility to approximate the zero accurate to four decimal places.

75. $f(x) = x^3 + x - 1$

76. $f(x) = x^3 + 3x - 2$

77. $g(t) = 2 \cos t - 3t$

78. $h(\theta) = 1 + \theta - 3 \tan \theta$

In Exercises 79–82, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of c guaranteed by the theorem.

79. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

80. $f(x) = x^2 - 6x + 8, \quad [0, 3], \quad f(c) = 0$

81. $f(x) = x^3 - x^2 + x - 2, \quad [0, 3], \quad f(c) = 4$

82. $f(x) = \dfrac{x^2 + x}{x - 1}, \quad \left[\dfrac{5}{2}, 4\right], \quad f(c) = 6$

83. ***Volume*** Use the Intermediate Value Theorem to show that for all spheres with radii in the interval $[0, 5]$, there is one with a volume of 275 cubic centimeters.

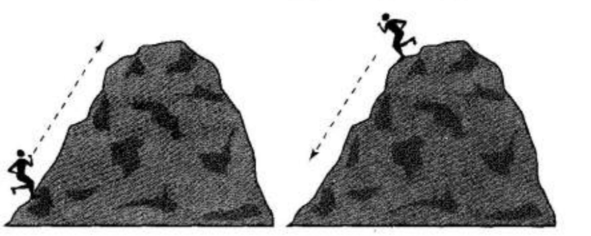
84. ***Telephone Charges*** A dial-direct long distance call between two cities costs \$1.04 for the first 2 minutes and \$0.36 for each additional minute or fraction thereof. Use the greatest integer function to write the cost C of a call in terms of the time t (in minutes). Sketch a graph of this function and discuss its continuity.

85. ***Inventory Management*** The number of units in inventory in a small company is given by

$$N(t) = 25\left(2\left[\!\left[\frac{t + 2}{2}\right]\!\right] - t\right)$$

where t is the time in months. Sketch the graph of this function and discuss its continuity. How often must this company replenish its inventory?

86. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]



Saturday 8:00 A.M. Sunday 8:00 A.M.

87. Prove that if f is continuous and has no zeros on $[a, b]$, then either

$f(x) > 0$ for all x in $[a, b]$

or

$f(x) < 0$ for all x in $[a, b]$.

88. Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

is not continuous at any real number.

89. Show that the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ kx, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x = 0$. (Assume that k is any nonzero real number.)

90. The **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Sketch a graph of $\operatorname{sgn}(x)$ and find the following limits (if possible).

(a) $\lim\limits_{x \to 0^-} \operatorname{sgn}(x)$ (b) $\lim\limits_{x \to 0^+} \operatorname{sgn}(x)$ (c) $\lim\limits_{x \to 0} \operatorname{sgn}(x)$

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If $\lim\limits_{x \to c} f(x) = L$ and $f(c) = L$, then f is continuous at c.

92. If $f(x) = g(x)$ for $x \neq c$ and $f(c) \neq g(c)$, then either f or g is not continuous at c.

93. A rational function can have infinitely many x-values at which it is not continuous.

94. The function $f(x) = |x - 1|/(x - 1)$ is continuous on $(-\infty, \infty)$.

95. ***Modeling Data*** After an object falls for t seconds, the speed S (in feet per second) of the object is recorded in the the table.

t	0	5	10	15	20	25	30
S	0	48.2	53.5	55.2	55.9	56.2	56.3

(a) Create a line graph of the data.

(b) Does there appear to be a limiting speed of the object? If there is a limiting speed, identify a possible cause.

96. Prove that if $\lim\limits_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0)$, then f is continuous at x_0.

97. Prove that for any real number y there exists x in $(-\pi/2, \pi/2)$ such that $\tan x = y$.

98. Discuss the continuity of the function $h(x) = x[\![x]\!]$.

99. Let $f(x) = \left(\sqrt{x + c^2} - c\right)/x$, $c > 0$. What is the domain of f? How can you define f at $x = 0$ in order for f to be continuous there?

100. Let $f_1(x)$ and $f_2(x)$ be continuous on the closed interval $[a, b]$. If $f_1(a) < f_2(a)$ and $f_1(b) > f_2(b)$, prove that there exists c between a and b such that $f_1(c) = f_2(c)$.