TECHNOLOGY Some graphing utilities such as *Derive, Maple, Mathcad, Mathematica,* and the *TI-92* perform symbolic differentiation. Others perform *numerical differentiation* by finding values of derivatives using the formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

where h is a small number such as 0.001. Can you see any problems with this definition? For instance, using this definition, what is the value of the derivative of $f(x) = |x|$ when $x = 0$?

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}\lim_{x\to c}\,[f(x) - f(c)] &= \lim_{x\to c}\left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\ &= \left[\lim_{x\to c}(x - c)\right]\left[\lim_{x\to c}\frac{f(x) - f(c)}{x - c}\right] \\ &= (0)[f'(c)] \\ &= 0\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. Therefore, f is continuous at $x = c$.

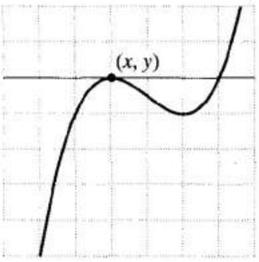
You can summarize the relationship between continuity and differentiability as follows.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. Thus, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and *not* be differentiable at $x = c$. Thus, continuity does not imply differentiability.

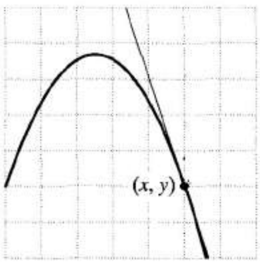
EXERCISES FOR SECTION 2.1

In Exercises 1 and 2, estimate the slope of the curve at the point (x, y).

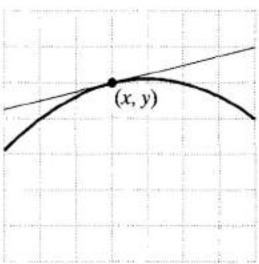
1. (a)



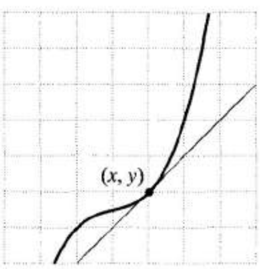
(b)



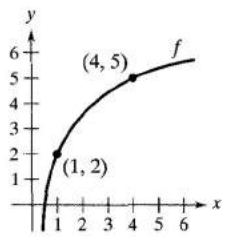
2. (a)



(b)



Think About It **In Exercises 3 and 4, use the graph shown in the figure.**



3. Identify or sketch each of the quantities on the figure.

(a) $f(1)$ and $f(4)$ (b) $f(4) - f(1)$

(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

4. Insert the proper inequality symbol (< or >) between the given quantities.

(a) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad \dfrac{f(4) - f(3)}{4 - 3}$

(b) $\dfrac{f(4) - f(1)}{4 - 1} \quad \square \quad f'(1)$

In Exercises 5–16, use the definition of the derivative to find $f'(x)$.

5. $f(x) = 3$
6. $f(x) = 3x + 2$
7. $f(x) = -5x$
8. $f(x) = 9 - \frac{1}{2}x$
9. $f(x) = 2x^2 + x - 1$
10. $f(x) = 1 - x^2$
11. $f(x) = x^3 - 12x$
12. $f(x) = x^3 + x^2$
13. $f(x) = \dfrac{1}{x - 1}$
14. $f(x) = \dfrac{1}{x^2}$
15. $f(x) = \sqrt{x - 4}$
16. $f(x) = \dfrac{1}{\sqrt{x}}$

In Exercises 17–22, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

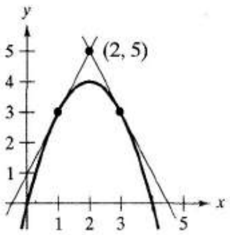
17. $f(x) = x^2 + 1$, $(2, 5)$
18. $f(x) = x^2 + 2x + 1$, $(-3, 4)$
19. $f(x) = x^3$, $(2, 8)$
20. $f(x) = \sqrt{x}$, $(1, 1)$
21. $f(x) = x + \dfrac{1}{x}$, $(1, 2)$
22. $f(x) = \dfrac{1}{x + 1}$, $(0, 1)$

In Exercises 23 and 24, find an equation of the line that is tangent to the graph of f *and* parallel to the given line.

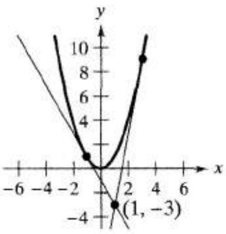
	Function	*Line*
23.	$f(x) = x^3$	$3x - y + 1 = 0$
24.	$f(x) = \dfrac{1}{\sqrt{x}}$	$x + 2y - 6 = 0$

In Exercises 25 and 26, find the equations of the two tangent lines to the graph of f that pass through the indicated point.

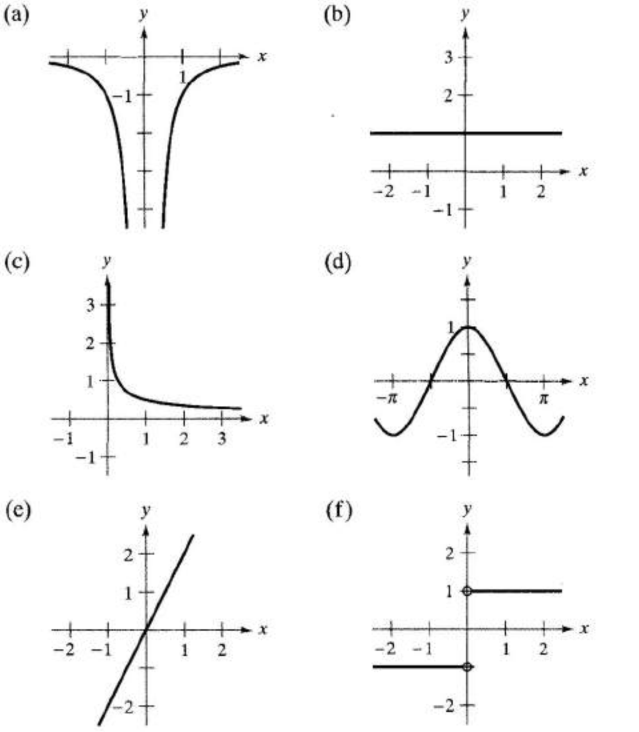
25. $f(x) = 4x - x^2$



26. $f(x) = x^2$

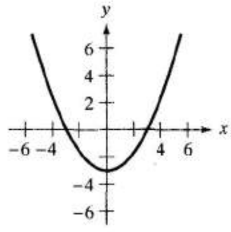


In Exercises 27–32, use your knowledge of the graph of the function and the geometric interpretation of the derivative to match the function with the graph of its derivative. It is not necessary to find the derivative of the function analytically.



27. $f(x) = x$
28. $f(x) = x^2$
29. $f(x) = \sqrt{x}$
30. $f(x) = \dfrac{1}{x}$
31. $f(x) = |x|$
32. $f(x) = \sin x$

33. ***Graphical Reasoning*** The figure shows the graph of g'.



(a) $g'(0) = \square$

(b) $g'(3) = \square$

(c) What can you conclude about the graph of g knowing that $g'(1) = -\frac{8}{3}$?

(d) What can you conclude about the graph of g knowing that $g'(-4) = \frac{7}{3}$?

(e) Is $g(6) - g(4)$ positive or negative? Explain.

(f) Is it possible to find $g(2)$ from the graph? Explain.

34. ***Graphical Reasoning*** Use a graphing utility to graph each function and its tangent lines when $x = -1$, $x = 0$, and $x = 1$. Based on the results, determine whether the slope of a tangent line to the graph of a function is always distinct for different values of x.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

35. ***Think About It*** Sketch a graph of a function whose derivative is always negative.

36. ***Think About It*** Sketch a graph of a function whose derivative is always positive.

Graphical, Numerical, and Analytic Analysis **In Exercises 37 and 38, use a graphing utility to graph f on the interval $[-2, 2]$. Complete the table by *graphically* estimating the slopes of the graph at the indicated points. Then evaluate the slopes *analytically* and compare your results with those obtained graphically.**

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

37. $f(x) = \frac{1}{4}x^3$ 38. $f(x) = \frac{1}{2}x^2$

Graphical Reasoning **In Exercises 39 and 40, use a graphing utility to graph the functions f and g in the same viewing rectangle where**

$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}.$$

Label the graphs and describe the relationship between them.

39. $f(x) = 2x - x^2$ 40. $f(x) = 3\sqrt{x}$

Technology **In Exercises 41 and 42, use a graphing utility to graph the function and its derivative in the same viewing rectangle. Label the graphs and describe the relationship between them.**

41. $f(x) = \dfrac{1}{\sqrt{x}}$ 42. $f(x) = \dfrac{x^3}{4} - 3x$

Writing **In Exercises 43 and 44, consider the functions $f(x)$ and $S_{\Delta x}(x)$ where**

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2).$$

(a) Use a graphing utility to graph f and $S_{\Delta x}$ in the same viewing rectangle for $\Delta x = 1, 0.5$, and 0.1.

(b) Give a written description of the graphs of S for the different values of Δx in part (a).

43. $f(x) = 4 - (x - 3)^2$ 44. $f(x) = x + \dfrac{1}{x}$

In Exercises 45–50, use the alternative form of the derivative to find the derivative at $x = c$ (if it exists).

45. $f(x) = x^2 - 1$, $c = 2$

46. $f(x) = x^3 + 2x$, $c = 1$

47. $f(x) = x^3 + 2x^2 + 1$, $c = -2$

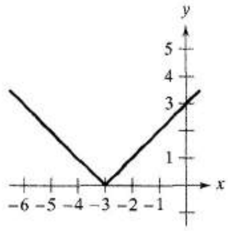
48. $f(x) = 1/x$, $c = 3$

49. $f(x) = (x - 1)^{2/3}$, $c = 1$

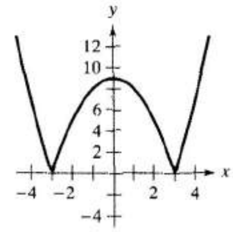
50. $f(x) = |x - 2|$, $c = 2$

In Exercises 51–60, describe the x-values at which f is differentiable.

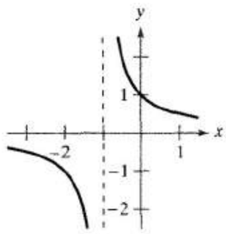
51. $f(x) = |x + 3|$



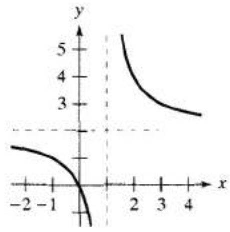
52. $f(x) = |x^2 - 9|$



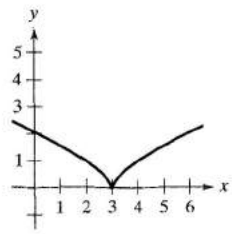
53. $f(x) = \dfrac{1}{x + 1}$



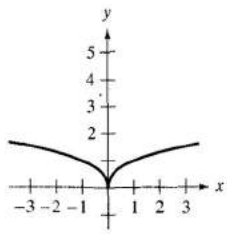
54. $f(x) = \dfrac{2x}{x - 1}$



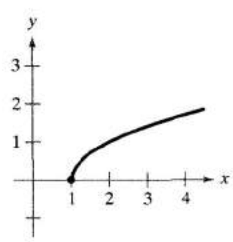
55. $f(x) = (x - 3)^{2/3}$



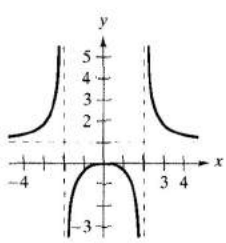
56. $f(x) = x^{2/5}$



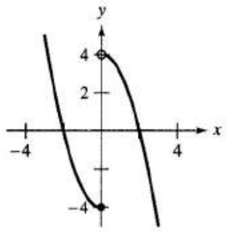
57. $f(x) = \sqrt{x - 1}$



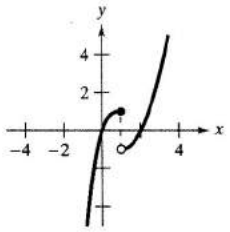
58. $f(x) = \dfrac{x^2}{x^2 - 4}$



59. $f(x) = \begin{cases} 4 - x^2, & x > 0 \\ x^2 - 4, & x \le 0 \end{cases}$



60. $f(x) = \begin{cases} x^2 - 2x, & x > 1 \\ x^3 - 3x^2 + 3x, & x \le 1 \end{cases}$



In Exercises 61–64, find the derivatives from the left and from the right at $x = 1$ (if they exist). Is the function differentiable at $x = 1$?

61. $f(x) = |x - 1|$

62. $f(x) = \sqrt{1 - x^2}$

63. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

64. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

In Exercises 65 and 66, determine whether the function is differentiable at $x = 2$.

65. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

66. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

67. ***Graphical Reasoning*** A line with slope m passes through the point $(0, 4)$ and has the equation $y = mx + 4$.

(a) Write the distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the function d in part (a). Based on the graph, is the function differentiable at every value of m? If not, where is it not differentiable?

68. ***Think About It*** Assume that $f'(c) = 3$. Find $f'(-c)$ given the following conditions.

(a) f is an odd function.

(b) f is an even function.

True or False? **In Exercises 69–71, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. If a function is continuous at a point, then it is differentiable at that point.

70. If a function has derivatives from both the right and the left at a point, then it is differentiable at that point.

71. If a function is differentiable at a point, then it is continuous at that point.

72. ***Conjecture*** Consider the functions $f(x) = x^2$ and $g(x) = x^3$.

(a) Graph f and f' on the same set of axes.

(b) Graph g and g' on the same set of axes.

(c) Identify any pattern between the functions f and g and their respective derivatives. Use the pattern to make a conjecture about $h'(x)$ if $h(x) = x^n$, where n is an integer and $n \ge 2$.

(d) Find $f'(x)$ if $f(x) = x^4$. Compare the result with the conjecture in part (c). Is this a proof of your conjecture? Explain.

73. Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Show that f is continuous, but not differentiable, at $x = 0$. Show that g is differentiable at 0, and find $g'(0)$.

74. ***Writing*** Use a graphing utility to graph the two functions $f(x) = x^2 + 1$ and $g(x) = |x| + 1$ in the same viewing rectangle. Use the zoom and trace features to analyze the graphs near the point $(0, 1)$. What do you observe? Which function is differentiable at this point? Write a short paragraph describing the geometric significance of differentiability at a point.