We conclude this section with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule.

Summary of Differentiation Rules

General Differentiation Rules

Let u and v be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}[cu] = cu'$$

Sum or Difference Rule:

$$\frac{d}{dx}[u \pm v] = u' \pm v'$$

Product Rule:

$$\frac{d}{dx}[uv] = uv' + vu'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

Simple Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\,u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

STUDY TIP As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

EXERCISES FOR SECTION 2.4

In Exercises 1–6, complete the table using Example 2 as a model.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^2 - 1}$		
4. $y = \tan(\pi x + 1)$		
5. $y = \csc^3 x$		
6. $y = \cos \dfrac{3x}{2}$		

In Exercises 7–30, find the first derivative of the algebraic function.

7. $y = (2x - 7)^3$
8. $y = (3x^2 + 1)^4$
9. $g(x) = 3(4 - 9x)^4$
10. $f(x) = 2(1 - x^2)^3$
11. $f(x) = (9 - x^2)^{2/3}$
12. $f(t) = (9t + 2)^{2/3}$
13. $f(t) = \sqrt{1 - t}$
14. $g(x) = \sqrt{3 - 2x}$
15. $y = \sqrt[3]{9x^2 + 4}$
16. $g(x) = \sqrt{x^2 - 2x + 1}$
17. $y = 2\sqrt{4 - x^2}$
18. $f(x) = -3\sqrt[4]{2 - 9x}$
19. $y = \dfrac{1}{x - 2}$
20. $s(t) = \dfrac{1}{t^2 + 3t - 1}$
21. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$
22. $y = -\dfrac{4}{(t + 2)^2}$

23. $y = \dfrac{1}{\sqrt{x+2}}$

24. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

25. $f(x) = x^2(x-2)^4$

26. $f(x) = x(3x-9)^3$

27. $y = x\sqrt{1-x^2}$

28. $y = x^2\sqrt{9-x^2}$

29. $y = \dfrac{x}{\sqrt{x^2+1}}$

30. $y = \dfrac{x^2}{\sqrt{x^9+9}}$

In Exercises 31–40, use a symbolic differentiation utility to find the first derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

31. $y = \dfrac{\sqrt{x}+1}{x^2+1}$

32. $y = \sqrt{\dfrac{2x}{x+1}}$

33. $g(t) = \dfrac{3t^2}{\sqrt{t^2+2t-1}}$

34. $f(x) = \sqrt{x}(2-x)^2$

35. $y = \sqrt{\dfrac{x+1}{x}}$

36. $y = (t^2-9)\sqrt{t+2}$

37. $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

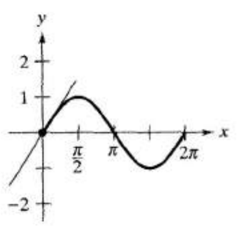
38. $g(x) = \sqrt{x-1} + \sqrt{x+1}$

39. $y = \dfrac{\cos \pi x + 1}{x}$

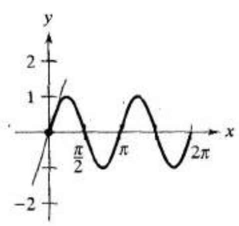
40. $y = x^2 \tan \dfrac{1}{x}$

In Exercises 41 and 42, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$.

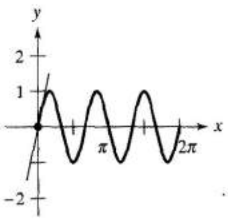
41. (a) $y = \sin x$



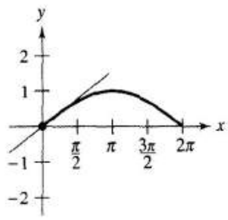
(b) $y = \sin 2x$



42. (a) $y = \sin 3x$



(b) $y = \sin \dfrac{x}{2}$



In Exercises 43–52, find the first derivative of the function.

43. $y = \cos 3x$

44. $y = \sin \pi x$

45. $g(x) = 3 \tan 4x$

46. $h(x) = \sec x^2$

47. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

48. $g(t) = 5\cos^2 \pi t$

49. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

50. $y = 3x - 5\cos(\pi x)^2$

51. $y = \sin(\cos x)$

52. $y = \sin\sqrt{x} + \sqrt{\sin x}$

In Exercises 53–60, evaluate the derivative of the function at the indicated point. You can use a graphing utility to verify your result.

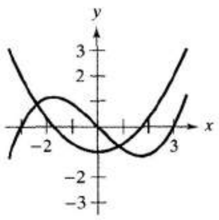
	Function	*Point*
53.	$s(t) = \sqrt{t^2+2t+8}$	$(2, 4)$
54.	$y = \sqrt[5]{3x^3+4x}$	$(2, 2)$
55.	$f(x) = \dfrac{3}{x^3-4}$	$\left(-1, -\dfrac{3}{5}\right)$
56.	$f(x) = \dfrac{1}{(x^2-3x)^2}$	$\left(4, \dfrac{1}{16}\right)$
57.	$f(t) = \dfrac{3t+2}{t-1}$	$(0, -2)$
58.	$f(x) = \dfrac{x+1}{2x-3}$	$(2, 3)$
59.	$y = 37 - \sec^3(2x)$	$(0, 36)$
60.	$y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 61–64, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

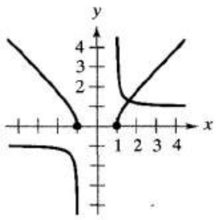
	Function	*Point*
61.	$f(x) = \sqrt{3x^2-2}$	$(3, 5)$
62.	$f(x) = \frac{1}{3}x\sqrt{x^2+5}$	$(2, 2)$
63.	$f(x) = \sin 2x$	$(\pi, 0)$
64.	$f(x) = \tan^2 x$	$\left(\dfrac{\pi}{4}, 1\right)$

Writing **In Exercises 65–68, the graphs of a function f and its derivative f' are given. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection.**

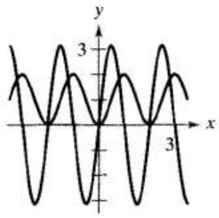
65.



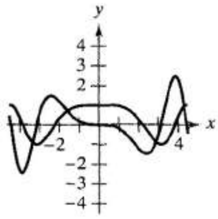
66.



67.



68.



In Exercises 69–72, find the second derivative of the function.

69. $f(x) = 2(x^2 - 1)^3$

70. $f(x) = \dfrac{1}{x - 2}$

71. $f(x) = \sin x^2$

72. $f(x) = \sec^2 \pi x$

73. ***Think About It*** Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

74. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that $f'(x) = g'(x)$.

75. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 m/s (use $-v$).

(b) the fire truck is moving away at a velocity of 30 m/s (use $+v$).

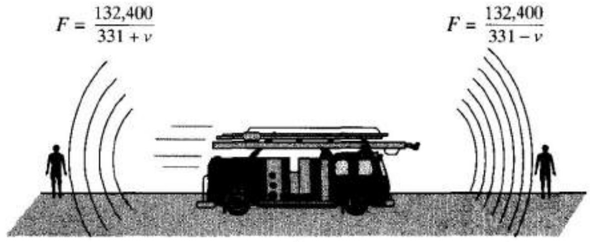


Figure for 75

76. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3} \cos 12t - \tfrac{1}{4} \sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

77. ***Pendulum*** A 15-centimeter pendulum moves according to the equation

$$\theta = 0.2 \cos 8t$$

where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

78. ***Wave Motion*** A buoy oscillates in simple harmonic motion

$$y = A \cos \omega t$$

as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

79. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

80. ***Modeling Data*** The normal daily maximum temperature T (in degrees Fahrenheit) for Denver, Colorado is given in the table. *(Source: National Oceanic and Atmosphere Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	43.2	46.6	52.2	61.8	70.8	81.4

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	88.2	85.8	76.9	66.3	52.5	44.5

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b\sin(\pi t/6 - c)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of temperature changes? Explain.

81. ***Think About It*** The table gives some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each of the following transformations of f.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

82. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results in part (a) to write general rules for the even- and odd-order derivatives

$f^{(2k)}(x)$ and $f^{(2k-1)}(x)$.

[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

83. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

84. Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

85. The geometric mean of x and $x + n$ is $g = \sqrt{x(x + n)}$, and the arithmetic mean is $a = [x + (x + n)]/2$. Show that

$$\frac{dg}{dx} = \frac{a}{g}.$$

86. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 87–90, use the result of Exercise 86 to find the derivative of the function.

87. $g(x) = |2x - 3|$ **88.** $f(x) = |x^2 - 4|$

89. $h(x) = |x| \cos x$ **90.** $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$P_1(x) = f'(a)(x - a) + f(a)$ **and**

$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$.

In Exercises 91 and 92, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

91. $f(x) = \sin\frac{x}{2}$, $a = \pi$ **92.** $f(x) = \sec 2x$, $a = \frac{\pi}{6}$

True or False? **In Exercises 93–95, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

93. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

94. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

95. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$