THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

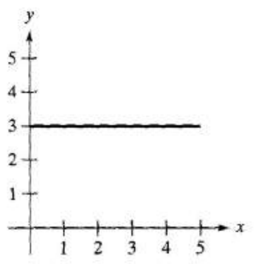
2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

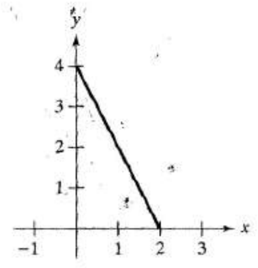
EXERCISES FOR SECTION 4.3

In Exercises 1–10, set up a definite integral that yields the area of the given region. (Do not evaluate the integral.)

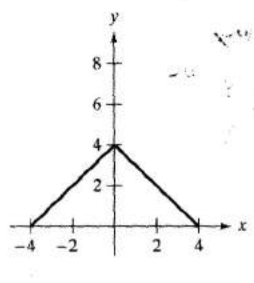
1. $f(x) = 3$



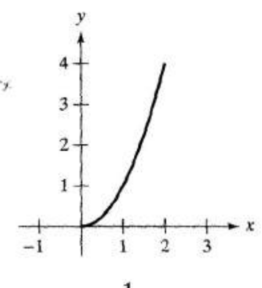
2. $f(x) = 4 - 2x$



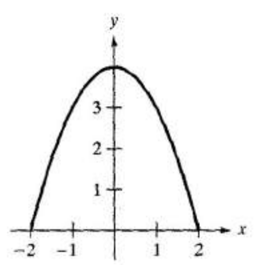
3. $f(x) = 4 - |x|$



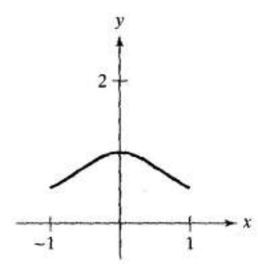
4. $f(x) = x^2$



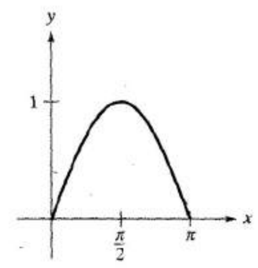
5. $f(x) = 4 - x^2$



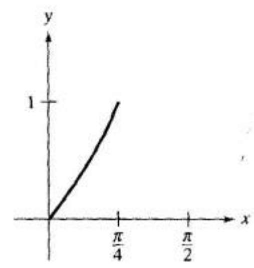
6. $f(x) = \dfrac{1}{x^2 + 1}$



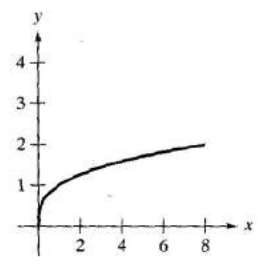
7. $f(x) = \sin x$



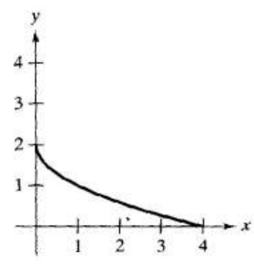
8. $f(x) = \tan x$



9. $g(y) = y^3$



10. $f(y) = (y - 2)^2$



In Exercises 11–20, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral ($a > 0, r > 0$).

11. $\displaystyle\int_0^3 4\,dx$

12. $\displaystyle\int_{-a}^a 4\,dx$

13. $\displaystyle\int_0^4 x\,dx$

14. $\displaystyle\int_0^4 \frac{x}{2}\,dx$

15. $\displaystyle\int_0^2 (2x + 5)\,dx$

16. $\displaystyle\int_0^5 (5 - x)\,dx$

17. $\displaystyle\int_{-1}^1 (1 - |x|)\,dx$

18. $\displaystyle\int_{-a}^a (a - |x|)\,dx$

19. $\displaystyle\int_{-3}^3 \sqrt{9 - x^2}\,dx$

20. $\displaystyle\int_{-r}^r \sqrt{r^2 - x^2}\,dx$

21. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find

(a) $\displaystyle\int_0^7 f(x)\,dx.$ (b) $\displaystyle\int_5^0 f(x)\,dx.$

(c) $\displaystyle\int_5^5 f(x)\,dx.$ (d) $\displaystyle\int_0^5 3f(x)\,dx.$

22. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, find

(a) $\displaystyle\int_0^6 f(x)\,dx.$ (b) $\displaystyle\int_6^3 f(x)\,dx.$

(c) $\displaystyle\int_3^3 f(x)\,dx.$ (d) $\displaystyle\int_3^6 -5f(x)\,dx.$

23. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find

(a) $\displaystyle\int_2^6 [f(x) + g(x)]\,dx.$ (b) $\displaystyle\int_2^6 [g(x) - f(x)]\,dx.$

(c) $\displaystyle\int_2^6 2g(x)\,dx.$ (d) $\displaystyle\int_2^6 3f(x)\,dx.$

24. Given $\int_{-1}^1 f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, find

(a) $\displaystyle\int_{-1}^0 f(x)\,dx.$ (b) $\displaystyle\int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx.$

(c) $\displaystyle\int_{-1}^1 3f(x)\,dx.$ (d) $\displaystyle\int_0^1 3f(x)\,dx.$

In Exercises 25–30, evaluate the definite integral by the limit definition.

25. $\displaystyle\int_4^{10} 6\,dx$ 26. $\displaystyle\int_{-2}^3 x\,dx$

27. $\displaystyle\int_{-1}^1 x^3\,dx$ 28. $\displaystyle\int_0^1 x^3\,dx$

29. $\displaystyle\int_1^2 (x^2 + 1)\,dx$ 30. $\displaystyle\int_1^2 4x^2\,dx$

In Exercises 31–34, express the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

Limit	*Interval*
31. $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
32. $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
33. $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
34. $\displaystyle\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

Write a program for your graphing utility to approximate a definite integral using the Riemann sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

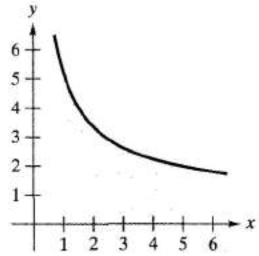
where the subintervals are of equal width. The output should give three approximations of the integral where c_i is the left-hand endpoint $L(n)$, midpoint $M(n)$, and right-hand endpoint $R(n)$ of each subinterval. In Exercises 35–38, use the program to approximate the definite integral and complete the table.

n	4	8	12	16	20
$L(n)$					
$M(n)$					
$R(n)$					

35. $\displaystyle\int_0^3 x\sqrt{3 - x}\,dx$ 36. $\displaystyle\int_0^3 \frac{5}{x^2 + 1}\,dx$

37. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx$ 38. $\displaystyle\int_0^3 x \sin x\,dx$

Think About It **In Exercises 39–42, use the figure to fill in the blank with the symbol <, >, or =.**



39. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \_\_\_\_ \quad \int_1^5 f(x)\,dx$$

40. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \_\_\_\_ \quad \int_1^5 f(x)\,dx$$

41. The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the midpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \_\_\_\_ \quad \int_1^5 f(x)\,dx$$

42. Let T be the average of the results of Exercises 39 and 40.

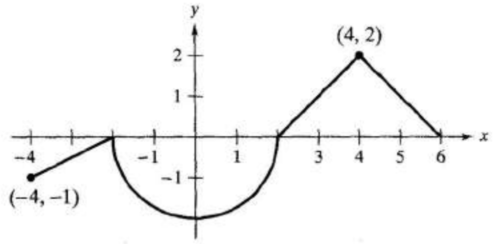
$$T \quad \_\_\_\_ \quad \int_1^5 f(x)\,dx$$

43. *Think About It* The graph of f below consists of line segments and a semicircle. Evaluate each definite integral by using geometric formulas.

(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$

(c) $\int_{-4}^2 f(x)\,dx$ (d) $\int_{-4}^6 f(x)\,dx$

(e) $\int_{-4}^6 |f(x)|\,dx$ (f) $\int_{-4}^6 [f(x)+2]\,dx$



44. *Think About It* Consider the function f that is continuous on the interval $[-5, 5]$ and for which $\int_0^5 f(x)\,dx = 4$. Evaluate each integral.

(a) $\int_0^5 [f(x)+2]\,dx$ (b) $\int_{-2}^3 f(x+2)\,dx$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even.) (d) $\int_{-5}^5 f(x)\,dx$ (f is odd.)

In Exercises 45 and 46, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

45. $\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) -3 (c) 10 (d) 2 (e) 8

46. $\int_0^{1/2} 4\cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) -6

True or False? **In Exercises 47–52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

48. $\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

49. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

50. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

51. The value of $\int_a^b f(x)\,dx$ must be positive.

52. If $\int_a^b f(x)\,dx > 0$, then f is nonnegative for all x in $[a, b]$.

53. Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$, where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 7$, and $x_4 = 8$, and where $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, and $c_4 = 8$.

54. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/3$, $x_3 = \pi$, and $x_4 = 2\pi$, and where $c_1 = \pi/6$, $c_2 = \pi/3$, $c_3 = 2\pi/3$, and $c_4 = 3\pi/2$.

In Exercises 55 and 56, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

55. $f(x) = \sqrt{x}$, $y = 0$, $x = 0$, $x = 2$
(*Hint:* Let $c_i = 2i^2/n^2$.)

56. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$
(*Hint:* Let $c_i = i^3/n^3$.)

57. *Think About It* Determine whether the function

$$f(x) = \frac{1}{x-4}$$

is integrable on the interval $[3, 5]$. Explain.

58. *Think About It* Determine whether the function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

59. *Think About It* Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

60. Evaluate, if possible, the integral $\int_0^2 [\![x]\!]\,dx$.

61. Determine $\lim_{x\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2]$ by using an appropriate Riemann sum.

62. Suppose f is integrable on $[a, b]$ and $m \le f(x) \le M$ for all x in $[a, b]$, $m > 0$, and $M > 0$. Prove that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Use the result to estimate $\int_0^1 \sqrt{1+x^4}\,dx$.

63. Prove that if f is a continuous function on a closed interval $[a, b]$, then

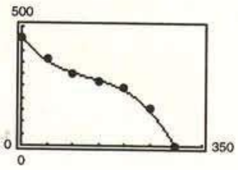
$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

69. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$A \approx 76{,}897.5\text{ ft}^2.$

(b)



Section 4.3 Riemann Sums and Definite Integrals

1. $\int_0^5 3\,dx$

2. $\int_0^2 (4 - 2x)\,dx$

3. $\int_{-4}^4 (4 - |x|)\,dx$

4. $\int_0^2 x^2\,dx$

5. $\int_{-2}^2 (4 - x^2)\,dx$

6. $\int_{-1}^1 \frac{1}{x^2 + 1}\,dx$

7. $\int_0^{\pi} \sin x\,dx$

8. $\int_0^{\pi/4} \tan x\,dx$

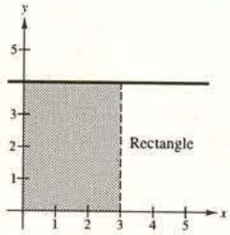
9. $\int_0^2 y^3\,dy$

10. $\int_0^2 (y - 2)^2\,dy$

11. Rectangle

$A = bh = 3(4)$

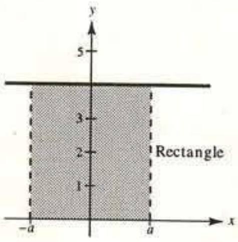
$A = \int_0^3 4\,dx = 12$



12. Rectangle

$A = bh = 2(4)(a)$

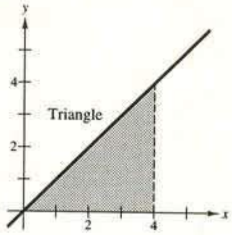
$A = \int_{-a}^{a} 4\,dx = 8a$



13. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$

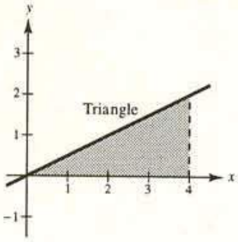
$A = \int_0^4 x\,dx = 8$



14. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

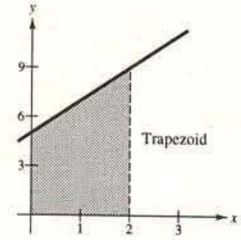
$A = \int_0^4 \frac{x}{2}\,dx = 4$



15. Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$

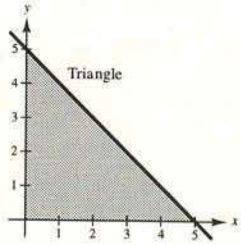
$A = \int_0^2 (2x + 5)\,dx = 14$



16. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(5)(5)$

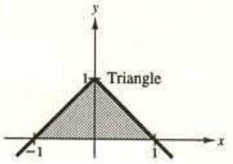
$A = \int_0^5 (5 - x)\,dx = \frac{25}{2}$



17. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

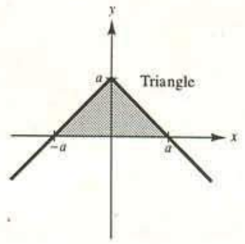
$$A = \int_{-1}^{1} (1 - |x|)\,dx = 1$$



18. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$$

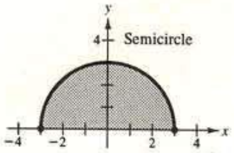
$$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$$



19. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

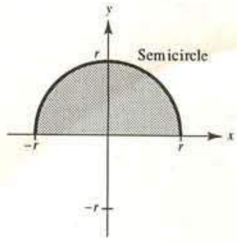
$$A = \int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$$



20. Semicircle

$$A = \frac{1}{2}\pi r^2$$

$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$$



21. (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

22. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

23. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$
$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$
$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

24. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

25. $y = 6$ on $[4, 10]$. $\left(\textbf{Note: } \Delta x = \frac{10 - 4}{n} = \frac{6}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = 36$$

$$\int_4^{10} 6\,dx = \lim_{n\to\infty} 36 = 36$$

43. (a) Quarter circle below x-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

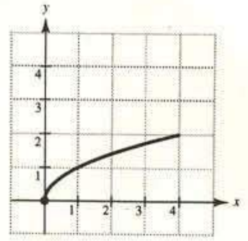
44. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$

(b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

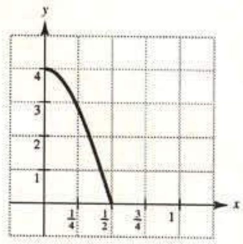
(d) $\int_{-5}^5 f(x)\,dx = 0$ (f odd)

45.



a. $A \approx 5$ square units

46.



b. $A \approx \frac{4}{3}$ square units

47. True

48. False

$$\int_0^1 x\sqrt{x}\,dx \neq \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

49. True

50. True

51. False

$$\int_0^2 (-x)\,dx = -2$$

52. False

$$\int_{-2}^4 x\,dx = 6$$

53. $f(x) = x^2 + 3x$, $[0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\sum_{i=1}^4 f(c_i)\,\Delta x = f(1)\,\Delta x_1 + f(2)\,\Delta x_2 + f(5)\,\Delta x_3 + f(8)\,\Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$