

Find any critical numbers of the function.

1a) $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Critical numbers: $x = 0, 2$

1b) $h(x) = \sin^2 x + \cos x$

$$0 < x < 2\pi$$

$$h'(x) = 2\sin x \cos x - \sin x$$

$$0 = 2\sin x \cos x - \sin x$$

$$0 = \sin x \cdot (2\cos x - 1)$$

$$\sin x = 0 \text{ or } 2\cos x - 1 = 0$$

$$\sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$x = \pi \text{ or } x = \frac{\pi}{3}, \frac{5\pi}{3}$$

1c) $f(x) = \frac{4x^2}{x+2}$

$$f'(x) = \frac{(x+2)8x - 4x^2 \cdot 1}{(x+2)^2}$$

$$f'(x) = \frac{8x^2 + 16x - 4x^2}{(x+2)^2}$$

$$f'(x) = \frac{4x^2 + 16x}{(x+2)^2}$$

$x = -2$ make $f'(x)$ undef

$$0 = 4x^2 + 16x$$

$$0 = 4x \cdot (x+4)$$

$$4x = 0 \quad x+4 = 0$$

$$x = 0 \text{ or } x = -4$$

Show that the Extreme Value Theorem can be applied, then locate the absolute extrema of the function on the closed interval.

2a) $g(x) = x^2 - 2x, [0, 4]$

Continuous polynomial on a closed interval so it can be applied!

$$g'(x) = 2x - 2 = 2(x-1)$$

Critical number: $x = 1$

Left endpoint: $(0, 0)$

Critical number: $(1, -1)$ Minimum

Right endpoint: $(4, 8)$ Maximum

2b) $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

Continuous poly. on closed interval, so yes

Left endpoint $f(-1) = -\frac{5}{2}$

Right endpoint $f(2) = 2$

$$f'(x) = 3x^2 - 3x$$

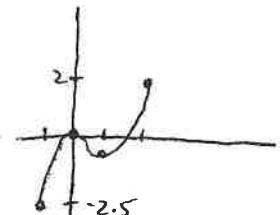
$$0 = 3x^2 - 3x$$

$$0 = 3x \cdot (x-1)$$

$$3x = 0 \quad x-1 = 0$$

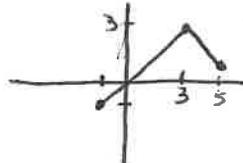
$$x = 0 \quad x = 1$$

$$f(0) = 0 \quad f(1) = -\frac{1}{2}$$



MAX $(2, 2)$

MIN $(-1, -\frac{5}{2})$



2c) $y = 3x^{2/3} - 2x, [-1, 1]$

EVT applies as "y" is continuous everywhere

2d) $f(x) = -|x-3| + 3, [-1, 5]$

EVT applies as $f(x)$ is continuous everywhere

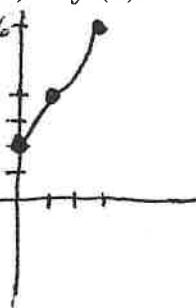
$$f(-1) = 5 \quad f(1) = 1 \quad f(0) = 0 \quad f(1) = 1 \quad f(-1) = -1 \quad f(5) = 1 \quad f(3) = 3$$

$$f'(x) = 2x^{-1/3} - 2 \quad \text{or} \quad \frac{2}{\sqrt[3]{x}} - 2$$

$f'(x)$ undefined at $x=0$

$$0 = \frac{2}{\sqrt[3]{x}} - 2 \quad 2\sqrt[3]{x} = 2 \quad \sqrt[3]{x} = 1 \quad x = 1$$

$$2e) \quad f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}, \quad [0, 3]$$



$f(x)$ continuous on $[0, 3]$ as $\lim_{x \rightarrow 1} f(x) = f(1)$

$f'(x)$ undefined at $x=1$

$$f'(x) = \begin{cases} 2, & 0 < x < 1 \\ 8x, & 1 < x < 3 \end{cases}$$

$f'(x)$ undefined at $x=3$

$$f'(x) = \begin{cases} -1, & x > 3 \\ 1, & x < 3 \end{cases}$$

$f'(x)$ never equal zero

$$\boxed{\begin{array}{l} \text{MIN } (-1, -1) \\ \text{MAX } (3, 3) \end{array}}$$

$$f(0) = 2$$

$$f(3) = 36$$

$$f(1) = 4$$

$$\boxed{\begin{array}{l} \text{MIN } (0, 2) \\ \text{MAX } (3, 36) \end{array}}$$

$f'(x)$ never equal zero

Determine whether Rolle's Theorem can be applied to on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c)=0$. If Rolle's Theorem cannot be applied, explain why not.

3a) $f(x) = -x^2 + 3x, \quad [0, 3]$

$$f(0) = f(3) = 0$$

f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$$f'(x) = -2x + 3$$

$$-2x + 3 = 0 \Rightarrow x = \frac{3}{2}$$

$$c\text{-value: } \frac{3}{2}$$

3b) $f(x) = \frac{x^2 - 2x - 3}{x + 2}, \quad [-1, 3]$

$f(x)$ undefined at $x=-2$ but not on the interval $[-1, 3]$

$$f'(x) = \frac{(x+2)(2x-2) - (x^2 - 2x - 3) \cdot 1}{(x+2)^2}$$

$$f'(x) = \frac{x^2 + 4x - 1}{(x+2)^2}$$

$f'(x)$ undefined at $x=-2$ but not on the interval $[-1, 3]$

so... continuous & differentiable } Rolles
Theorem
Applies
 $f(-1) = 0 \quad f(3) = 0$

NEED $0 = \frac{x^2 + 4x - 1}{x+2}$

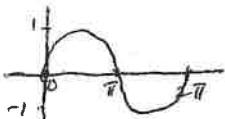
QUAD
Formula
or
G.C.

$$0 = x^2 + 4x - 1$$

$$x = \frac{-4 \pm \sqrt{25}}{2}$$

$$\boxed{x = -2 + \sqrt{5}}$$

or
 ≈ -0.236



3c) $f(x) = \sin x, [0, 2\pi]$

$f(x)$ continuous & differentiable $f'(x)$ is continuous

$f(0) = 0 \quad f(2\pi) = 0$

so it applies

$f'(x) = \cos x$

$0 = \cos x$

$x = \frac{\pi}{2} + \frac{3\pi}{2}$



3d) $f(x) = (x-1)^{2/3}, [-2, 2]$

$f(x)$ NOT differentiable at $x=1$ which is in the interval AND

$f(-2) = -3^{2/3} \quad f(2) = 1$

Rolle's Theorem does NOT apply

3e) $f(x) = (x-3)(x+1)^2, [-1, 3]$

$f(x)$ is continuous & differentiable

$f(-1) = 0 \quad f(3) = 0$ so it applies

$f'(x) = (x-3) \cdot 2(x+1) + (x+1)^2$

$0 = 2(x-3)(x+1) + (x+1)^2$

$0 = (x+1)(2x-6+x+1)$

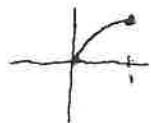
$0 = (x+1)(3x-5)$

$x=-1$

$\textcircled{x=5/3}$

$x+1=0 \quad x=-1$
 $3x-5=0 \quad x=5/3$

Determine whether the Mean Value Theorem can be applied to on the closed interval $[a, b]$. If the MVT can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. If the MVT cannot be applied, explain why not.



4a) $f(x) = x^2, [-2, 1]$

$f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$

$$x = -\frac{1}{2}$$

$$c = -\frac{1}{2}$$

4b) $f(x) = x^{2/3}, [0, 1]$

$f(x)$ is continuous and differentiable on $(0, 1)$. Note that it is NOT differentiable at $x=0$, but is at any value greater than that

$f(0) = 0 \quad f(1) = 1$

Avg ROC = $\frac{1-0}{1-0} = 1$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

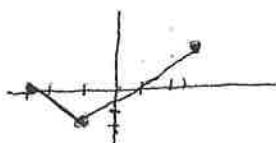
$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$

$$1 = \frac{2}{3\sqrt[3]{x}}$$

$$\sqrt[3]{x} = 2$$

$$(\sqrt[3]{x})^3 = (\frac{2}{3})^3$$

$\textcircled{x = \frac{8}{27}}$



4c) $f(x) = |x+1| - 2, [-3, 3]$

$f(x)$ is continuous

$f(x)$ NOT differentiable at $x = -1$ which is on the interval.

So... MVT does not apply

5) AP MULTIPLE CHOICE EXAMPLES

- 1) If f is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$, which of the following could be false?

MVT (A) $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

(B) $f'(c) = 0$ for some c such that $a < c < b$.

ONLY TRUE IF $f(a) = f(b)$
Rolle's Theorem

EVT (C) f has a minimum value on $a \leq x \leq b$.

EVT (D) f has a maximum value on $a \leq x \leq b$.

- 2) Let f be the function given by $f(x) = x^3 - 3x^2$. What are all values of c that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval $[0, 3]$?

(A) 0 only

(B) 2 only

(C) 3 only

(D) 0 and 3

(E) 2 and 3

$f(x)$ continuous & differentiable

(0, 0) AVG ROC

$$(3, 0) \quad \frac{0}{3} = 0$$

$$f'(x) = 3x^2 - 6x$$

$$\text{AVG ROC} = \text{INST ROC}$$

$$0 = 3x^2 - 6x$$

$$0 = 3x(x-2)$$

$$3x=0 \quad x-2=0$$

$$x=0 \quad x=2$$

NOTE: we are guaranteed a value on $(0, 3)$
means inside interval
 $x=0$ is NOT inside the interval

INSIDE THE INTERVAL

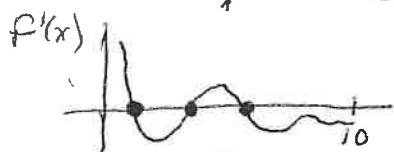
3) GRAPHING CALCULATOR ALLOWED

The first derivative of the function f is given by $f'(x) = \frac{\cos^2 x}{x} - \frac{1}{5}$. How many critical values does f have on the open interval $(0, 10)$?

- (A) One
- (B) Three**
- (C) Four
- (D) Five
- (E) Seven

$f'(x)$ undefined when $x = 0$

$$0 = \frac{\cos^2 x}{x} - \frac{1}{5}$$

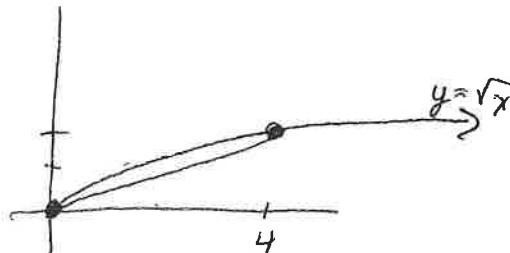


but interval
DOES NOT
include
that x -valu

Three times $f'(x) = 0$

- 4) The Mean Value Theorem guarantees the existence of a special point on the graph of $y = \sqrt{x}$ between $(0, 0)$ and $(4, 2)$. What are the coordinates of this point?

- (A) $(2, 1)$
- (B) $(1, 1)$**
- (C) $(2, \sqrt{2})$
- (D) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$
- (E) None of the above



AVG RUC $\frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{2\sqrt{x}}$$

$$2\sqrt{x} = 2$$

$$y(1) = 1$$

$$\sqrt{x} = 1$$

$$x = 1$$

$$y = x^{1/2}$$

$$y' = \frac{1}{2}x^{-1/2}$$

$$y' = \frac{1}{2\sqrt{x}}$$